Topology in Condensed Matter
Summary of Lectures

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1 Useful Notation

iff	if and only if
$\Rightarrow$	if then
$\equiv$	defined as
$\therefore$	therefore
$\because$	because
$\square$	end of proof
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbb{I}$	identity matrix
$\forall$	the universal quantifier, for all
$\exists$	the existential quantifier, there exists
$\in$	is an element of
$\subset$	is a subset of
$\cup$	union of sets
$\cap$	intersection of sets

2 Some Mathematical Background in Topology

- Let be any set and $\mathcal{T} = \{U_i | i \in I\}$ denote a specific collection of subsets of X. The pair $(X, \mathcal{T})$ is a topological space if $\mathcal{T}$ satisfies the following requirements:

  (i) $0, X \in \mathcal{T}$

  (ii) If J is any subcollection of I the family $\{U_j | j \in J\}$ satisfies $\cup_{k \in K} U_k \in \mathcal{T}$

  (iii) If K is any finite subcollection of I the family $\{U_k | k \in K\}$ satisfies $\cap_{k \in K} U_k \in \mathcal{T}$

  X itself is often called a topological space. The U_i are the open sets and $\mathcal{T}$ gives a topology to X.

- Let X and Y be topological spaces. A map $f : X \to Y$ is continuous if the inverse image of open set in Y is an open set in X.

- Let X_1 and X_2 be topological spaces. A map $f : X_1 \to X_2$ is a homeomorphism if it is continuous and has an inverse $f^{-1} : X_2 \to X_1$ which is also continuous. If there exists a homeomorphism between X_1 and X_2, X_1 is said to be homeomorphic to X_2 and vice versa. If two topological spaces have different topological spaces then they are not homeomorphic to each other.

- Let X_1 and X_2 be topological spaces. X_1 and X_2 have the same homotopy type if there exists a map $f : X_1 \to X_2$ and a map $g : X_2 \to X_1$.

3 Quantum Phase Transitions, Topological Order, and Long Range Entanglement

- Let $H(d)$ be a local Hamiltonian with d some parameter and $|\psi_0(d)\rangle$ its ground state. We are interested in gapped systems in which all excitations above the ground state have a finite energy in the thermodynamic limit. For a local observable $\hat{P}$ we define its expectation value as

$$P(d) = \langle\psi_0(d)|\hat{P}|\psi_0(d)\rangle\,.$$

  We consider a quantum phase transition to occur for $H(d)$ at $d = d_c$ when there is a singularity in $P(d)$ at $d = d_c$. States $|\psi_0(a)\rangle$ and $|\psi_0(b)\rangle$ belong to the same phase if there is a smooth path $[a, b]$ on which no quantum phase transition occurs.

  - If a local Hamiltonian $H(d)$ has a gap for all d on the path $[a, b]$ then there is no phase transition along the path.
  - If $|\psi_0(a)\rangle$ and $|\psi_0(b)\rangle$ are both gapped and in the same phase, then there is a path $[a, b]$ on which the gap is always non-zero.

- Iff two gapped ground states are in the same phase can they be adiabatically connected by a local unitary evolution:

$$|\psi_0(a)\rangle \sim |\psi_0(b)\rangle \text{ iff } \exists\tilde{H}(d) : |\psi_0(b)\rangle\mathcal{T}_d\,\mathrm{e}^{-\mathrm{i}\int_a^b \mathrm{d}\delta\tilde{H}(\delta)}\,|\psi_0(a)\rangle$$

  $\mathcal{T}_d$ is ordering along the contour $[a, b]$ in the d parameter space and $\tilde{H}(d)$ is a local Hermitian operator.

- A state has short range order iff it can be transformed to a direct product state via a local unitary transformation. Such states are said to have trivial topological order.

- A state which cannot be transformed into a direct product state via a local unitary transformation is said to have long-range entanglement, i.e. non-trivial topological order. Topological order defines the equivalence classes defined by local unitary evolution.