Due: before the end of the semester. If you have any questions or confusion, please just contact me.

1. The Su-Schrieffer-Heeger (SSH) model is one of the simplest examples of a topological insulator. It describes a chain of atoms with alternating hopping strengths, originally introduced to describe systems like polyacetylene (one also finds it as the electronic part resulting from a Peierl's instability). It can be written as
$$H = -t\sum_j c_j^\dagger c_{j+1}\left(1+\delta\mathrm{e}^{\mathrm{i}\pi j}\right) + \mathrm{H.c.}\,.$$
$c_j^\dagger$ is a spinless fermionic creation operator on a site j. Schematically this looks like figure 1. Rewrite this Hamiltonian using Pauli matrices where the basis consists of the two atoms in the unit cell (coloured green and blue).

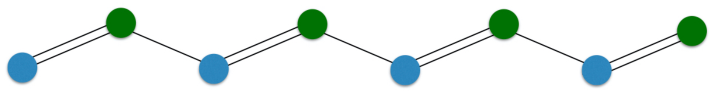

Figure 1: The SSH chain. The different hopping strengths between sites are drawn with one or two lines.

2. By considering a chain of an even number of atoms $2N$, and periodic boundary conditions, make the discrete transformation
$$c_j = \frac{1}{\sqrt{N}}\mathrm{e}^{\mathrm{i}kj}c_k$$
to find the Hamiltonian in momentum space. Writing this as
$$H = \sum_k c_k^\dagger c_k \vec{d}_k\cdot\vec{\boldsymbol{\sigma}}$$
find $\vec{d}_k$. Calculate the topological invariant (assuming that k is a continuous variable). When is this system topologically nontrivial? Sketch the band structure (i.e. the energy as a function of crystal momentum k) and show that the gap closes when the system changes from topologically nontrivial to trivial.

3. When the system is topologically nontrivial we expect that there are protected edge states present in the finite system. This model has in fact a semi-analytical solution in this case [1]. However instead of using that we will find the edge state solutions to a low energy continuum version. These edge states will have zero energy.

   (a) Starting from the momentum space Hamiltonian make a Taylor expansion around the minimum of the band structure k^*. Keep only lowest order terms. You should have a Hamiltonian with a linear dispersion in $q = k - k^*$ and mass terms.

   (b) Write the continuum Hamiltonian in real space by replacing $q \to -\mathrm{i}\partial_x$.

   (c) The finite length of the chain can be mimicked by writing the mass term as positive and divergent at $x = \pm L/2$. Note that you will need to take the limit of these divergent mass terms $m(x = \pm L/2) \to \infty$ at the appropriate moment to find the boundary conditions. Find the zero energy solutions to this equation by solving Schrödinger's equation with the energy explicitly zero. When do you have edge solutions? What is their density profile? How does this compare with the topological invariant you already found?

[1] Byeong Chun Shin. A formula for Eigenpairs of certain symmetric tridiagonal matrices. *Bulletin of the Australian Mathematical Society*, 55:249–249, 1997.