1. A quantum violin string can vibrate at frequencies ω, 2ω, 3ω and so on. Each vibration mode can be treated as an independent harmonic oscillator. Ignoring the zero point energy, each mode with frequency $p\omega$ has energy $E = n\hbar p\omega$, $n = 0, 1, 2, \ldots$. Write an expression for the average energy of the string at temperature T. Show that at large temperatures the free energy is given by

$$F = \frac{\pi^2 k_B^2 T^2}{6\hbar\omega} .$$

(Tip: Riemann zeta function gives $\zeta(2) = \pi^2/6$, which can be written as an integral.)

2. Find an expression for the heat capacity at constant volume in terms of the free energy $F = E - TS$.

3. Find the free energy for an ultrarelativistic ideal gas of particles where $E = c|\vec{p}|$. Using the expression from the previous question find the specific heat capacity.

$$\zeta(s) = \sum_{p=1}^{\infty} \frac{1}{p^s} = \frac{\int_0^\infty dx \frac{x^{s-1}}{e^x - 1}}{\int_0^\infty dx x^{s-1} e^{-x}} .$$