Preliminaries:

- Inverse temperature: $\beta = 1/k_B T$.
- Canonical ensemble: $\rho(E_n) = e^{-\beta E_n}/Z$.
- Partition function for the canonical ensemble: $Z = \sum_n e^{-\beta E_n}$.

1. Prove Stirling's formula.

   (i) First show that

   $$N! = \int_0^\infty e^{-x} x^N dx = \int_0^\infty e^{-g(x)} dx \,.$$

   What is $g(x)$?

   (ii) Approximate $g(x) \approx g(x_0) + g(x_0)(x - x_0)2/2$ where x_0 is the location of the minima of $g(x)$.

   (iii) Show that $N! \approx \sqrt{2\pi N} N^N e^{-N}$ (you will need one further approximation).

   (iv) What is the accuracy of Stirlings formula for the small value of $N = 5$?

2. (i) Show that two coupled systems in the microcanonical ensemble maximize their entropy at equal temperature only if the heat capacity is positive.

   (ii) In the canonical ensemble, show that the fluctuations in energy $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$ are proportional to the heat capacity.

   (iii) Show that in the canonical ensemble the entropy can be written as $S = k_B \partial_T (T \ln Z)$.

3. Consider a system consisting of N spin-$\frac{1}{2}$ particles, each of which can be in one of two quantum states, up and down. In a magnetic field B, the energy of a spin in the up/down state is $\pm \mu B/2$ where μ is the magnetic moment.

   (i) Show that the partition function is

   $$Z = 2^N \cosh^N \frac{\beta \mu B}{2} \,.$$

   (ii) Find the average energy E and entropy S.

   (iii) Check that your results for both quantities make sense at $T = 0$ and in the limit $T \to \infty$.