Let us return to our SSH model which we have previously dealt with, and which we have also seen in the graphene zig-zag edges:

$$H = -t(1+\delta)\sum_j \Psi_j^\dagger \sigma^x \Psi_j - t(1-\delta)\left[\sum_j \Psi_{j+1}^\dagger \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Psi_j + \text{H.c}\right] . \tag{1}$$

For periodic boundary conditions we can make the Fourier transform $\Psi_j = \frac{1}{\sqrt{N}}\sum_k e^{ikj}\Psi_k$ where $k = 2\pi n/N$ with $n = 1, 2, \dots N$ and N the number of unit cells. Then

$$H = \sum_k \Psi_k^\dagger \underbrace{\begin{pmatrix} 0 & \sigma_k^* \\ \sigma_k & 0 \end{pmatrix}}_{\equiv \mathcal{H}_k} \Psi_k \, , \tag{2}$$

where $\sigma_k = -t(1+g) - t(1-g)e^{-ik}$.

1. Find the two non-unitary symmetries of this Hamiltonian: $\{S_1, \mathcal{H}_k\}_+ = 0$ and $[S_2, \mathcal{H}_k]_- = 0$. Together these give a unitary symmetry $\{S, \mathcal{H}_k\}_+ = 0$ with $S = S_1 S_2$. What are S_1^2, S_2^2, and S^2?

2. A topological invariant can be found by rewriting the Hamiltonian as $\mathcal{H}_k = \vec{d}_k \cdot \vec{\sigma}_k$. What do the symmetries mean for the vector $\vec{d}_k$. By considering the behaviour of $\vec{d}_k$ for $k : 0 \to 2\pi$ find a topological invariant.

3. In one dimension the topological invariant we are interested in is equivalent to the winding number or Zak-Berry phase ν. The Zak-Berry phase for a single negative energy band μ is

$$\nu = \frac{\varphi}{2\pi} = i \int_0^{2\pi} \frac{dk}{2\pi} \langle u_k | \partial_k u_k \rangle \, , \tag{3}$$

with the integral taken round the Brillouin zone and $|u_k\rangle$ being the eigenfunction of a negative energy band. This results in a $\mathbb{Z}$ invariant. The number of pairs of topologically protected edge states is then equal to the total winding number for all negative energy bands. Calculate ν.