1. Consider the following one dimensional Hamiltonian:

$$\hat{H}|\psi\rangle = -J\sum_{j=1}^{N}\left[(1+\delta e^{i\pi j})\hat{c}_j^\dagger \hat{c}_{j+1} + \text{H.c.}\right]|\psi\rangle = \epsilon|\psi\rangle\,.$$

This describes a chain of atoms located at sites $j = 1, 2, \ldots N$. What does the term $c_j^\dagger c_{j+1}$ mean? We assume that the operators are fermionic $\{\hat{c}_j, \hat{c}_l^\dagger\} = \delta_{j,l}$ etc, and the chain is periodic so that site $N+1$ is the same as site 1, i.e. $\hat{c}_1 = \hat{c}_{N+1}$.

   (a) To solve such a Hamiltonian we want to diagonalise it, i.e. we wish to find a Hamiltonian like $\hat{H} = \sum_k \hat{\epsilon}_k c_k^\dagger \hat{c}_k$. Why does this help and what is ϵ_k?

   (b) For this new Hamiltonian to make sense we need that $\{\hat{c}_k, \hat{c}_q^\dagger\} = \delta_{k,q}$. For the transform

   $$c_j = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{ikj} c_k$$

   show that this is true.

   (c) Find the Hamiltonian in terms of these new operators and diagonalise it.

2. Consider the following one dimensional Hamiltonian:

$$\hat{H} = \sum_{k\sigma}\xi_k \hat{c}_{k\sigma}^\dagger \hat{c}_{k\sigma} + \sum_k\left[\Delta_k \hat{c}_{k\uparrow}^\dagger \hat{c}_{-k\downarrow}^\dagger + \Delta_k \hat{c}_{-k\downarrow}\hat{c}_{k\uparrow}\right]\,.$$

This is the BCS Hamiltonian for superconductivity. We can diagonalise this Hamiltonian using a transform like $\hat{f}_{k0}^\dagger = \bar{u}_k \hat{c}_{k\uparrow}^\dagger - \bar{v}_k \hat{c}_{-k\downarrow}$ and $\hat{f}_{k1}^\dagger = \bar{u}_k \hat{c}_{-k\downarrow}^\dagger + \bar{v}_k \hat{c}_{k\uparrow}$.

   (a) Show $\hat{f}_{k0,1}$ obey canonical commutation relations and find the conditions on u and v that allow this.

   (b) Diagonalise the Hamiltonian by first inverting these relations for the $\hat{c_{k\sigma}}$ operators. Find expressions for u_k and v_k.

   (c) Write $\hat{H} = \hat{\Psi}_k^\dagger \mathcal{H}_k \hat{\Psi}_k$ for a suitable vector of operators $\hat{\Psi}_k$ and find the eigenvalues, compare with the answer from before.