THE COULOMB BLOCKADE IN QUANTUM DOTS AND A METAMAGNETIC QUANTUM CRITICAL POINT

by

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Abstract

This thesis is concerned with two different phenomena, related by the use of similar techniques. Using Keldysh non-equilibrium field theory we have studied the Coulomb blockade regime in closed quantum dots and the metamagnetic quantum critical point in $Sr_3Ru_2O_7$. In the first of these we have found some previously unknown behaviour of the tunnelling density of states of the dot at the degeneracy point and its effect on the conductance. The second consists of analysing phenomenological field theories for the system.

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Table of Common Abbreviations and Useful Definitions

The Boltzmann constant	$k_B = 1$
The Planck constant	$\hbar = 1$
Free particle density of states (dimension d)	ν_d
4-vector notation: position and time	$x = (\mathbf{r}, t)$
4-vector notation: momentum and frequency	$Q = (\mathbf{q}, \omega)$
Electron charge	e
Charging energy of a quantum dot	$E_c = \frac{e^2}{2C}$ (C is capacitance)
Inverse temperature	$\beta = T^{-1}$
Elastic lifetime	τ
Diffusion coefficient	$D = \frac{v_F^2 \tau}{d}$
Inelastic lifetime	τ_{in}
Thouless length	$L_{Th} = \sqrt{D\tau_{in}}$
Real part of z	$\Re z$
Imaginary part of z	$\Im z$
Free particle dispersion	$\xi_{\mathbf{q}} = \varepsilon_{\mathbf{q}} - \mu = \frac{\mathbf{q}^2}{2m} - \mu$
Pauli Matrices	$\boldsymbol{\tau}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	$\boldsymbol{\tau}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
	$\boldsymbol{\tau}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
	$\boldsymbol{\tau}^0 = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Chapter 1

INTRODUCTION

1.1 Structure of the Thesis

This thesis is divided into two sections on two different subjects, though they are united by the techniques we use to analyse them. The first subject is the tunnelling density of states and conductance across many electron almost closed quantum dots. First we present an overview of the area of study which leads to our interest in the problem of the Coulomb blockade and then give a qualitative explanation of the phenomenon. After this we explain the necessary theoretical tools and then present our original work in chapter 4. This consists of analysing the structure of the saddle point solution to correctly account for the Coulomb blockade and to derive the tunnelling density of states and differential conductance of the system. The second topic is metamagnetic quantum criticality, an introduction to which is left to chapter 5 preceding our work on these systems. Here we are interested in a variation to the phenomenological Hertz-Millis field theory used to describe these systems. We present some corrections to the outcome of this model and look at the possibility of how this model may be justified. This model and the quantities calculated from it are then generalized to the Keldysh technique. Note that for simplicity we use $\hbar = k_B = 1$ throughout.

1.2 The Effect of Interactions and Disorder

Mesoscopics deals with the area in which the electrons phase is coherent over large lengths. This phase coherence is destroyed by inelastic collisions. This gives us a length scale, the

Thouless length L_{Th}, over which the electron is coherent. We shall look at this idea presently. We are interested in systems where $L_{Th} \sim L$, the system size. Because of this property interference effects can become very important. Typically, for the systems we are interested in, the mean free path between elastic collisions, l, satisfies the following conditions: $l \gg \lambda_F$, the Fermi wavelength, and $l \ll L$, the system size.

This area has been a rich source of theoretical work since the late seventies, and advances in fabrication methods and experimental techniques have allowed a wealth of data to be collected on mesoscopic systems. We shall focus only on a few issues that are relevant to the work we present in this thesis. The effect of interactions and disorder on a system will be described. We are interested in its application to the zero bias anomaly in quantum dots, this then leads to our main focus on the Coulomb blockade.

One of the main contributions of mesoscopics has been the joint role of interaction and disorder and how this contributes to a break down of the Landau picture of a Fermi metal. Introducing disorder into a system will break the translational invariance over some length scale, and hence momentum conservation starts to break down. This means that the interaction can no longer be taken care of by the rescaling of some parameters, as in the Landau picture, which relies on momentum being a good quantum number[1].

The perturbative techniques developed calculate the corrections to the free particle picture caused by disorder and interaction. (In the regime where we have the small parameters $\varepsilon\tau \ll 1$ and $(k_F l)^{-1} \ll 1$. l and τ are, respectively, the mean free path between elastic collisions and the associated lifetime.)

For a weakly disordered system it is possible to describe processes in terms of scattering from impurities. The waves which are scattering are the electron Bloch waves of a perfect crystal structure. But this view neglects to take into account the effects of interaction. And, as mentioned above, with the addition of disorder to a system neglecting electron-electron interactions becomes an increasingly invalid assumption. In other words they are no longer adequately taken into account by the Landau picture of a Fermi metal[2].

Viewing disorder as the scattering of Bloch waves will give a conductivity like $\sigma = \sigma_0 - A\sigma_0^2 T^n$[3], where σ_0 is the residual conductivity due to impurities in the sample, and we are at low temperatures. A and n are positive constants which depend upon the mechanism for resistance. For example it could be caused by electron-electron collisions or scattering from thermally activated phonons. However, even in the weak disorder limit this model is incorrect, and the correction from disorder to the Drude conductivity must consider disorder from the beginning.

In the limit of very strong disorder, Anderson discovered that the wavefunctions of the electrons can become localised[4]. This is when the wavefunctions, due to scattering processes, are no longer extended across the sample but have some localization length ξ. If disorder is strong enough then the localization length becomes small enough to spatially confine electrons. It can be shown that for arbitrarily weak disorder this will always be the case for one dimension. For two dimensions it is less well known what will occur. One important concept which is used in connection with this issue is the mobility edge. The mobility edge is defined as the energy at which states change from being localised to being extended. If the Fermi energy is in the localised region then there will be no conduction at zero temperature, $\sigma_0 = 0$. But extended states will insure there is a residual conductivity at zero temperature. There is a metal-insulator transition at the mobility edge as conduction becomes possible when the wavefunctions are sufficiently extended.

Let us consider static ($\omega = 0$) conductivity. From appendix A the expression for the cooperon contribution to conductivity is given by

$$\sigma = \underbrace{\frac{ne^2\tau}{m}}_{\sigma_0} - \frac{2e^2}{\pi}\frac{1}{L^d}\sum_Q \frac{1}{Q^2}. \tag{1.1}$$

(The cooperon will be explained in chapter 2.) σ_0 is the Drude conductivity. This shows the lowest order correction to the Drude conductivity from disorder effects. This sum can be performed for the different dimensions yielding

$$\sigma(L) = \begin{cases} \sigma_0 - \frac{e^2}{\pi^3}\left(\frac{1}{l} - \frac{1}{L}\right) & \text{if } d = 3, \\ \sigma_0 - \frac{e^2}{\pi^2}\ln\left(\frac{L}{l}\right) & \text{if } d = 2, \\ \sigma_0 - \frac{e^2}{\pi}(L - l) & \text{if } d = 1. \end{cases} \tag{1.2}$$

The upper and lower cut-offs on the sum over momentum Q are given by the inverse mean free path, l^{-1}, and system size, L^{-1}, respectively.

Inelastic collisions will occur between electrons. This is the origin of the timescale τ_{in}, which is the lifetime of an energy eigenstate of the random potential. If this is greater than the elastic scattering lifetime, $\tau \ll \tau_{in}$, then it gives an upper time that electrons can diffuse for. Correspondingly we have the Thouless length

$$L_{Th} = (D\tau_{in})^{\frac{1}{2}}. \tag{1.3}$$

This is the distance an electron will diffuse between the inelastic collisions which will cause dephasing. The diffusion constant is given by $D = v_F^2\tau/d$. Hence any scale dependent localization or interference effect has an upper cut-off of L_{Th}. The effective dimension of a sample is given by the number of dimensions for which the size of the system is greater than L_{Th}.

Now if we consider scattering as a temperature dependent mechanism we can write $L_{Th} = aT^{-\frac{p}{2}}$[3]. With arbitrary constants, a and $p > 0$, depending on the scattering mechanism. This is from a power law relation $\tau_{in} \propto T^{-p}$. Using this and redefining $l = aT_0^{-\frac{p}{2}}$ a temperature dependent conductivity correction is, for example in two dimensions,

$$\sigma(T) = \sigma_0 + \frac{p}{2}\frac{e^2}{\pi^2}\ln\Big(\frac{T}{T_0}\Big). \tag{1.4}$$

This is the so called weak localization correction. The conductivity decreases with decreasing T which is the sign of localization occurring. As T decreases L_{Th} increases, hence the scale over which quantum interference is effective increases. So localization becomes more evident and the conductivity decreases. The effect of disorder *and* interactions will be explained with respect to the zero bias anomaly.

1.3 Quantum Dots

A quantum dot is a system used to confine electrons. They are small enough to be considered zero dimensional, which in the language described before means they have a diameter less than L_{Th}. As shall be seen later the signature of such a structure is that its zero momentum mode is dominant, as might be naively expected. For example a lateral dot can be created in the two dimensional interface between gallium arsenide and aluminium gallium arsenide semiconductors. Chromium and gold layers are used to form

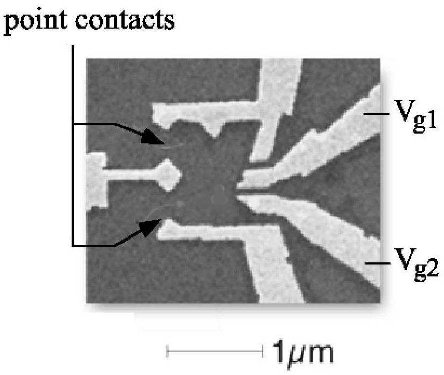


Figure 1.1: A scan of a quantum dot taken from[7]. The point contacts are coupled to the leads which allow electrons in and out of the dot. The electrons are confined in the central cavity, the size of which can be altered by varying the gate voltages V_{g1} and V_{g2}.

the necessary patterns for the gates and to apply a potential to form the depleted region of electrons. See figure 1.1 for a scanning micrograph of a quantum dot. Negative voltages are then applied to surface gates to move electrons in or out of the dot. For reviews of the properties of quantum dots see Kouwenhoven and Marcus[5] and Alhassid[6]. They can typically be from nanometers to micrometers in size and can confine from one to one thousand electrons. As seen in figure 1.1 this lateral type of quantum dot will be of the order of micrometers in diameter.

Confining electrons in this small space leads to the quantization of energy levels and charge. The two phenomena of quantum dots that we are interested in are the manifestations of the zero bias anomaly and the Coulomb blockade. These two effects refer to open and closed dots respectively. In an open dot it is easy to tunnel between the dot and the leads. A closed dot has only weak tunnelling between itself and any leads attached to it. As the dot is "closed" to the leads, charge becomes quantized inside and the Coulomb repulsion between electrons in the dot causes the charging energy to manifest itself.

We can write a general model for an isolated quantum dot. The Hamiltonian, with $\{i, j, k, l\}$ labelling the states and $\{\alpha, \beta\}$ as the spin labels, is

$$\hat{H} = \sum_{\substack{ij \\ \alpha}} \varepsilon_{ij} \hat{f}^{\dagger}_{i\alpha} \hat{f}_{j\alpha} + \frac{1}{2} \sum_{\substack{ijkl \\ \alpha\beta}} u_{ijkl} \hat{f}^{\dagger}_{i\alpha} \hat{f}^{\dagger}_{j\beta} \hat{f}_{k\beta} \hat{f}_{l\alpha} \tag{1.5}$$

u_{ijkl} is the interaction term defined in terms of the electron-electron interaction, $V(\mathbf{r})$, and the single particle states of the system, $\psi_i(\mathbf{r})$. We note that ε_{ij} is spin independent as we assume the energy states are spin degenerate. $\hat{f}^\dagger_{i\alpha}$ creates a particle in this state and we can also write

$$u_{ijkl} = \int d\mathbf{r}d\mathbf{r}'\psi_i^*(\mathbf{r})\psi_j^*(\mathbf{r}')V(\mathbf{r}-\mathbf{r}')\psi_k(\mathbf{r}')\psi_l(\mathbf{r}). \tag{1.6}$$

We will simplify this Hamiltonian to the version we use. Firstly we take the energy spectrum to consist of a randomly spaced set of levels with a mean level spacing δ which is small (c.f. temperature and all relevant energy scales.) Secondly we simplify the interaction term[8]. In the limit that the dimensionless conductance $g = E_T/\delta \sim \sqrt{N} \to \infty$ it is possible to neglect all off-diagonal terms in the interaction[9, 10]. The proof of this, however, is non-trivial and we shall not reproduce it here. (The Thouless energy is $E_T \sim v_F/L$ in the 2-d ballistic limit.) Neglecting off diagonal terms leaves us with only three possible terms which can contribute. Along with the one we consider below there is a spin interaction and a cooper interaction: these are neglected. The spin, or exchange, interaction is neglected as the mean level spacing is larger than the exchange energy, $\delta \gg E_s$. The exchange energy is the energy difference between electrons with parallel and anti-parallel spins[11]. Hence $\hat{H}_{int} = \frac{1}{2}E_c\hat{N}^2$, where $\hat{N}$ is the total number operator and $E_c = e^2/2C$ is the charging energy of the dot from the total capacitance C. In the terms of equation (1.6) we can write

$$E_c \sim \frac{1}{2}\int \frac{d^2\mathbf{r}}{L^2}V(\mathbf{r}) \tag{1.7}$$

for the charging energy, which trivially recovers the expression in terms of capacitance when we insert the Coulomb energy into the integral. The Hamiltonian thus becomes

$$\hat{H} - \mu\hat{N} = \sum_k \hat{f}^\dagger_k \xi_k \hat{f}_k + \frac{E_c}{2}\hat{N}^2. \tag{1.8}$$

We shall use this simplified Hamiltonian to analyse the Coulomb blockade. Though even this simplified form for the interaction cannot be dealt with perturbatively and we need to be more careful. The case in which spin becomes important has been investigated recently by Kisilev and Gefen[12].

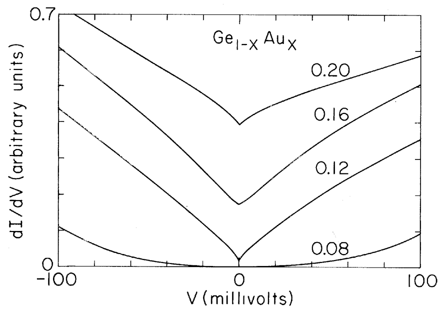


Figure 1.2: The differential conductance versus voltage for $Ge_{1-x}Au_x$. Values of x marked on data. From[17].

1.4 The Zero Bias Anomaly

The zero bias anomaly is the effect that, as the potential (or bias) across the dot is lowered, the differential conductance vanishes more quickly than expected. The differential conductance, G, vanishes as $|V|^{\frac{1}{2}}$ for low temperatures ($T < eV$)[13], compared with the Ohmic result: $G \sim$ constant. For example see figure 1.2. This was first explained by Altshuler and Aronov[14, 15] by the joint effects of disorder and interaction on the tunnelling density of states. This effect has been seen experimentally many times, for example Rowel[16] and McMillan and Mochel[17]. This effect is explained by the interplay of disorder and electron-electron interactions. A dip in the differential conductance will correspond to a gap forming in the single particle (tunnelling) density of states. They are related at zero temperature by the simple formula $\delta G(V)/G = \delta\nu(eV)/\nu$[13].

We present a qualitative explanation of the zero bias anomaly[18]. We present a proper calculation of the tunnelling density of states in chapter 3. In this case we are considering an almost open quantum dot, hence charging effects are neglected and we cannot explain the zero bias anomaly in terms of charging energy effects. The process can be split into two separate sections. Firstly an electron must cross the potential barrier, secondly it needs to spread across the metal (dot). For small bias it is the second term which will dominate, as it has become easier for the electron to cross the barrier. If the electron crosses the metal in a shorter time than the metal's relaxation time then the other electrons can be said to be approximately stationary. An electrostatic potential is set up across the system by the tunnelling electron and the hole left behind. Before the next electron can tunnel into this classically forbidden region, the charge (of the previous tunnelled electron) must diffuse across the system, until the electrostatic potential is lower than the bias voltage. This can take a long time and is what gives rise to the dip in conductivity at zero bias.

This is analogous to the Coulomb blockade where the conductance across the quantum dot is periodically suppressed. Both the Coulomb blockade and zero bias phenomena are related by the dominant effect being due to the zero momentum mode of the electron-electron interaction. We will discuss this in more detail in chapters 3 and 4.

1.5 The Coulomb Blockade

The Coulomb blockade is a phenomenon seen in almost closed quantum dots[6, 10]. In the dot the conductance is exponentially suppressed due to the charging energy associated with adding electrons. However, for periodic values of an applied gate voltage the conductance peaks. Historically, attention was first applied to the statistics of the positions and heights of these peaks[19, 20, 21, 22] and then to the tunnelling density of states of the quantum dot[23, 24, 25]. We are interested in the tunnelling density of states as this allows us to look at the connection between the zero bias anomaly and the Coulomb blockade.

The following is a simple explanation for the phenomenon of the Coulomb blockade. When the n^{th} electron enters the quantum dot it costs a charging energy to counter the Coulomb repulsion of $E_C(n) = n^2e^2/2C$. So the tunnelling of electrons through the dot will be suppressed unless

$$E_C(n+1) - (n+1)V_g = E_C(n) - nV_g. \tag{1.9}$$

With V_g as the gate voltage across the dot. This is satisfied by the condition $V_g = (n + \frac{1}{2})E_c$. The periodic suppression of tunnelling away from the values $V_g = (n + \frac{1}{2})E_c$ is the Coulomb blockade. We refer to the points at which n and $n + 1$ electrons in the dot have the same energy as degeneracy points. At these values it costs nothing to add an electron and hence there is a finite conductance. The data in figure 1.4 clearly shows the peaks and suppression of conductance at different gate voltages.

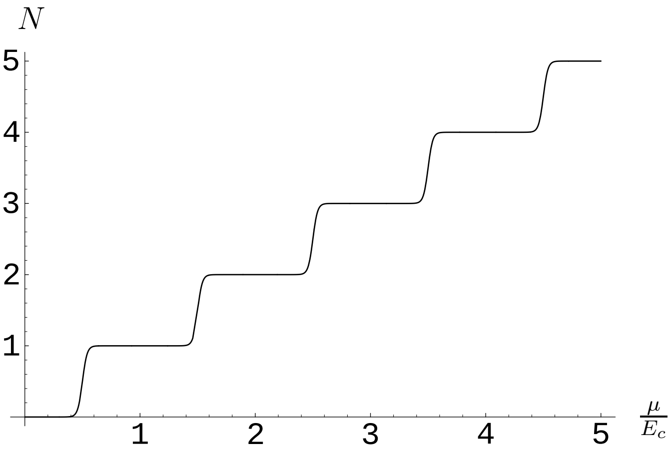


Figure 1.3: Coulomb Staircase for $T/E_c = 0.02$ from equation(4.35).

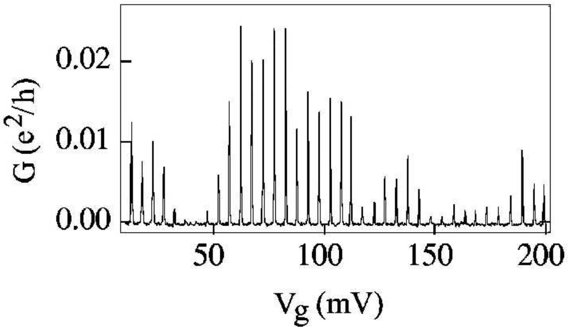


Figure 1.4: Differential conductance G in a quantum dot as a function of the gate voltage V_g. From[7].

From this analysis we gain the simple picture of the "Coulomb staircase". As you change V_g the number of particles on the dot will change by 1 as it passes through the degeneracy points. See figure 1.3.

The orthodox theory of the Coulomb blockade is due to Shekter and Khulik[19] and later Averin and Likharev[20]. It considers rate equations for tunnelling through a central dot weakly coupled to two leads. By considering Fermi's golden rule we can write an equation for the current into and out of the dot from the leads[8, 26]. Tunnelling between the energy levels ε_n in the lead and ξ_k in the dot, with amplitude t_n leads to a current between the left lead and the dot of

$$I_L = 2\pi e \sum_{kn} |t_{Ln}|^2 \delta(\xi_k + eV_L + E_N - E_{N+1} - \varepsilon_n)[P_N f(\xi_k)\{1 - f(\varepsilon_n)\} \\ - P_{N+1} f(\varepsilon_n)\{1 - f(\xi_k)\}]. \qquad (1.10)$$

E_N is the energy due to electrostatics of the state with N electrons and P_N is the probability of being in this state. We consider the leads to be reservoirs which are consequently always in equilibrium. Also the relaxation time for electrons which have tunnelled into the dot is considered to be short so that the distribution function in the dot is also a Fermi function, $f(\varepsilon)$, and we do not consider the relaxation process for the electrons once they have tunnelled into the dot. Note that as we are interested in the steady state scenario the current into and out of the dot must be equal. If, in equation (1.10), we integrate over the energy levels and demand that $P_N + P_{N+1} = 1$, so that the dot must have either N or $N + 1$ electrons then we find[8]:

$$G = \left.\frac{dI}{dV}\right|_{V=V_L-V_R=0} = e^2 \nu_0 \frac{\Gamma_L \Gamma_R}{\Gamma_L + \Gamma_R} \frac{2E_c\beta(N_0 - N_0^*)}{\sinh[2E_c\beta(N_0 - N_0^*)]}. \qquad (1.11)$$

The exponentially suppressed nature of the differential conductance is the phenomenon of the Coulomb blockade. $\Gamma_\alpha = 2\pi|t_\alpha|^2\nu_\alpha$ and $N_0 - N_0^*$ measures the distance from the degeneracy point where the current is at a peak. This would be altered by shifting a gate voltage applied to the dot.

Transport through quantum dots[27] can also be calculated from the tunnelling density of states[28, 29, 30]. We derive the standard formula in section 4.3. We are principally interested in this approach as we wish to analyse the tunnelling density of states. A gap in the tunnelling density of states away from the degeneracy point is the origin of the suppression of conductance (as there are no states available to tunnel into in the dot). The closing of this gap at the degeneracy point allows electrons to flow through the dot and produces a finite conductance. As we demonstrate in chapter 4, we find a full description of the form of the tunnelling density of states. Both at the degeneracy points and in the valleys. The correct form of the tunnelling density of states at the degeneracy point is required to retain the correct form of differential conductance in equation (1.11).

More recent work has focused on extending the description to the scenarios of granular arrays of dots and almost open quantum dots. Matveev[31] has described a dot strongly coupled to one bulk lead and calculates the average particle number on the dot with changing gate voltage. Some residual effect of charging is found. The average particle number, $\langle N \rangle$, periodically oscillates as the gate voltage, $\propto N$, is changed:

$$\langle N \rangle = N - C|r|\sin(2\pi N). \tag{1.12}$$

(The reflection amplitude $|r|$ satisfies $|r| \ll 1$ in the case of weak reflection, and C is a constant.) This effect is small however, as the dot is only weakly isolated from the lead. While it is evidently not possible to measure transport properties such as the conductance of such a system, it is not a purely theoretical consideration. The effect of such a system in close proximity to, and hence interacting with, another dot can be measured[32]. The conductance through this second dot, which can be coupled to two leads, can then be measured.

This coupled system of the dot and the lead is written entirely in terms of the lead electrons. This is done by using the fact that the current into the dot and out of the dot must be equal. This Hamiltonian is then bosonized and solved in perturbation theory recovering the Coulomb staircase, etc for almost perfect transmission across the barrier between dot and lead. This can be generalized to the case of two leads connected to a dot[33] and a granular system[34]. The granular system consists of a string of dots connected by macroscopic leads.

Kamenev and Gefen[24] relate the Coulomb blockade phenomenon to the zero bias anomaly. We shall consider the zero bias anomaly in more detail in chapter 3. They consider the tunnelling density of states using a functional integral Matsubara representation. While this recovers the suppression of the density of states in the Coulomb valleys their saddle point approximation misses the information associated with the Coulomb staircase.

Using the Matsubara technique they derive an expression for the Green's function. First a Hubbard-Stratonovich transformation is applied and then a gauge transformation is used to remove all non-zero Matsubara frequency modes for the Bosonic fields. The zero frequency mode cannot be removed however and is dealt with using a saddle point approximation.

The zero Matsubara frequency mode we need to deal with is:

$$\int d\phi_0 e^{-\frac{\beta\phi_0^2}{2E_c} - i\phi_0 N_0 + \Omega^0(\mu - i\phi_0)} G^0_\alpha(\tau_i - \tau_f, \mu - i\phi_0). \tag{1.13}$$

(N_0 is the local potential.) Hence the saddle point approximation is

$$0 = (\mu - \bar{\mu})/E_c + N_0 + \partial\Omega^0(\bar{\mu})/\partial\mu \text{ where} \tag{1.14}$$

$$\bar{\mu} = \mu - i\phi_0 \tag{1.15}$$

solves this equation. This is valid for $\beta E_c \ll 1$. Ω^0 is the free particle potential and $G^0(\tau_i - \tau_f)$ is the free particle Green's function. Hence it is possible to write

$$G_\alpha(\tau_i - \tau_f, \mu) = G^0_\alpha(\tau_i - \tau_f, \bar{\mu}) D(\tau_i - \tau_f) \tag{1.16}$$

$$D(\tau_i - \tau_f) = e^{-T\sum_{m\neq 0} \frac{E_c}{\omega_m^2}(1 - e^{i\omega_m(\tau_i - \tau_f)})} \tag{1.17}$$

From this they get

$$\frac{\nu(\varepsilon)}{\nu_0} = -\frac{1}{2}\int \frac{d\omega}{2\pi}\Big[\tanh[(\varepsilon - \omega)/2T] + \coth(\omega/2T)\Big] B(\omega) \tag{1.18}$$

$$B(\omega) = -2\Im D^R(\omega) = \frac{\sqrt{2\pi}}{\sqrt{TE_c}}\left(e^{-(\omega + E_c/2)^2/2E_cT} - e^{(\omega - E_c/2)^2/2E_cT}\right) \tag{1.19}$$

which they claim describes the gap in the tunnelling density of states. However this result is in fact only valid for temperatures $T \gg E_c$. We recover this result in the large temperature regime but for low temperatures, $T \ll E_c$ the density of states is different. The gap they see is therefore not the one giving rise to the Coulomb blockade but some high temperature remnant. From their saddle point approximation given above and equation (1.16) it is easy to see that the average number of particles is given by

$$\langle N \rangle = -\frac{\partial\Omega}{\partial\mu} = N_0 + \frac{\mu - \bar{\mu}}{E_c}. \tag{1.20}$$

It is linear with changing chemical potential rather than exhibiting the "staircase".

A more recent work is that of Efetov and Tschersich[35] who correctly describe the saddle point. They are interested in a general granular system but we shall ignore that aspect of their work and focus on a single grain. In a similar way to Kamenev and Gefen they use a Hubbard-Stratonovich transformation and gauge out all but the zero-frequency Bosonic fields. The important difference is the inclusion of the infinite number of winding numbers in the Gauge field used. This is directly analogous to the infinite number of saddle point solutions we use in our formalism.

After the Hubbard-Stratonovich transformation, where $\bar{N}$ is the local dimensionless potential, we have the action

$$\begin{aligned} S = {} & \frac{1}{4E_c}\int V^2(\tau)d\tau - i\bar{N}\int V(\tau)d\tau \\ & + \int \bar{\psi}_\tau[\partial_\tau - \xi + iV(\tau)]\psi_\tau d\tau \end{aligned} \tag{1.21}$$

integrated over the fields V, $\bar{\psi}$ and ψ. The Gauge transformation is

$$\psi(\tau) \to \psi(\tau)e^{-i\tilde{\phi}(\tau)} \tag{1.22}$$

which removes the field $V(\tau)$. The periodic function $\tilde{\phi}(\tau) = \phi(\tau) + 2\pi kT\tau$ and k are the winding numbers. $d_\tau\tilde{\phi}(\tau) = \tilde{V}(\tau)$ and $V(\tau) = \rho + 2\pi Tk + \tilde{V}(\tau)$ with $\int_0^\beta \tilde{V}(\tau)d\tau = 0$. The field ρ is the zero Matsubara frequency field which cannot be removed by the gauge transformation. This is evaluated in the same saddle point approximation as Kamenev and Gefen used in the regime where the mean level spacing $\delta \ll T$.

The tunnelling density of states is

$$\begin{aligned} \nu(\varepsilon) &= \Im[\tilde{\nu}(\varepsilon_n)|_{\varepsilon_n \to -i\varepsilon+\delta}] \\ \tilde{\nu}(\varepsilon_n) &= -(\pi)^{-1}\int d\tau e^{i\varepsilon_n\tau}\sum_\alpha G_\alpha(\tau) \end{aligned} \tag{1.23}$$

Evaluating the free particle Green's function leaves

$$\tilde{\nu}(\varepsilon_n) = \nu_0 T\int d\tau \frac{e^{i\varepsilon_n\tau}}{\sin(\pi T\tau)}\underbrace{\langle e^{-i[\tilde{\phi}(\tau)-\tilde{\phi}(0)]}\rangle}_{=\Pi(\tau)} \tag{1.24}$$

and we need to evaluate $\Pi(\tau)$. Separating the winding numbers from the Gauge field allows them to be summed separately in $\Pi(\tau)$. Also averaging over the fields yields

$$\Pi(\tau) = \langle e^{-i[\phi(\tau)-\phi(0)]} \rangle_\phi \langle e^{-2\pi i k \tau T} \rangle_k . \tag{1.25}$$

The actions for the averaging are

$$S_\phi = \frac{T}{4E_c} \operatorname{Tr} \phi_n \omega_n^2 \phi_{-n} \quad \text{and} \quad S_k = \frac{T\pi^2}{E_c} k^2 . \tag{1.26}$$

The average over ϕ is the same as for the action of Kamenev and Gefen and gives $e^{-B(\tau-\tau^2 T)}$. The sum over k is performed using the Poisson trick:

$$\sum_N e^{i\phi N} = \sum_m \delta(\phi - 2\pi m). \tag{1.27}$$

The result, with the necessary normalization, is

$$\sum_m e^{-\beta m E_c m + 2\tau E_c m - T\tau^2 E_c} . \tag{1.28}$$

These results are placed into the equation for the tunnelling density of states. Then, performing the τ integral and analytic continuation, we are left with the result

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{1}{Z} \sum_m e^{-m^2 E_c \beta} [f(\varepsilon + E_c - 2E_c m) + f(-\varepsilon + E_c + 2E_c m)]. \tag{1.29}$$

So far their approach does not include a gate voltage as for the granular system it would be an additional complication. The distribution of the potential across the granular system would need to be taken into account. However for our purposes, i.e. for a single grain, it is equivalent to introducing a potential N_0 by replacing $m \to m + N_0$. At the degeneracy point this will give a “half-gap” in the tunnelling density of states of a width $\sim E_c$. We shall show how to derive this in our method in chapter 4. However, away from the degeneracy point the tunnelling density of states is not correctly described. Whereas we can correctly describe the tunnelling density of states both at the peaks and the valleys of the Coulomb blockade.

1.6 Summary

In this chapter we have presented an overview of our approach to the phenomena of the Coulomb blockade. We have presented interaction and interference effects as an introduction to the zero bias anomaly presented in chapter 3. This effect is unified in its explanation with the main focus of the first part of this thesis: the Coulomb blockade. In chapter 4 we shall calculate the tunnelling density of states and conductance for a blockaded quantum dot. First, however we shall introduce the necessary technical tools we use throughout this thesis.

Chapter 2

KELDYSH GREEN'S FUNCTIONS AND FUNCTIONAL INTEGRALS

Firstly we will introduce the basic ideas and techniques of Green's functions and their representation as functional integrals. We introduce zero temperature and Keldysh nonequilibrium diagrammatic techniques and derive some basic results which we shall use. These include disorder averaging and Dyson's equation as well as the derivation of the polarisation operator and diffuson. We then introduce Grassman fields, coherent states and the functional integral representation for Green's functions.

2.1 Green's Functions

Green's functions can be used to calculate many properties of a system and the techniques involved will be used extensively in this thesis. Here we shall devote ourselves exclusively to Keldysh Green's functions and zero temperature Green's functions. We shall not discuss Matsubara representation. The zero temperature form is a useful introduction to the Keldysh form as Keldysh Green's functions use many of the same ideas. We will also used the zero temperature technique for comparisons in chapter 6. Before defining and elucidating the idea of Green's functions we briefly introduce the different representations of wavefunctions and operators that can be used in quantum mechanics which shall be useful here.

There are three representations of the wavefunctions and operators in quantum mechanics that we shall consider. They differ in where the time dependence resides: in the operators or in the wavefunctions. The Schrodinger representation is where all time dependence is in the wavefunctions and the operators acting on the system are time independent. The Heisenberg representation is where, conversely, all the time dependence is within the operators. It is a trivial operation to pass between these two, using

$$i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi, \text{ which formally gives } \Psi(t) = e^{-i\hat{H}t}\Psi(t = t_0) \tag{2.1}$$

$$\text{and therefore } \hat{\phi}_H(t) = e^{i\hat{H}t}\hat{\phi}_S e^{-i\hat{H}t}, \tag{2.2}$$

$\hat{H}$ is the Hamiltonian of the Schrodinger equation and $\hat{\phi}$ is a general operator. If the Hamiltonian is time dependent we replace $i\bar{H}t$ by $i\int^t \bar{H}(t)$. However more useful for our purposes than either of these is to separate the time dependence of the interacting and the free Hamiltonians, $\bar{H} = \bar{H}_0 + \bar{H}_i$. This is done in such a way that the operators will contain the free Hamiltonian time dependence but not the time dependence of the interaction.

$$\hat{\phi}_I(t) = S^{-1}(t)\hat{\phi}_H S(t) \tag{2.3}$$

$$S(t) = \mathrm{T}\, e^{-i\int_{-\infty}^{t} H_i(t')dt'} \tag{2.4}$$

$$\hat{\phi}_I(t) = e^{i\hat{H}_0 t}\hat{\phi}_S e^{-i\hat{H}_0 t}. \tag{2.5}$$

T is time ordering and is defined as

$$\mathrm{T}\, a(t_1)b(t_2) = \begin{cases} a(t_1)b(t_2) & \text{if } t_1 > t_2, \\ \mp b(t_2)a(t_1) & \text{if } t_1 < t_2, \end{cases} \tag{2.6}$$

and $\mp$ is for the fermionic/bosonic case.

2.1.1 Green's Functions at Zero Temperature

A Green's function can be thought of, in the simplest way, as the inverse of a differential operator, at least for the single particle case. But it is possible to write them for more general many particle and interacting systems. The single particle Green's function can be defined as the solution to

$$(\hat{\varepsilon} - \hat{H}_0)G(\mathbf{r}, t; \mathbf{r}', t') = [i\partial_t - \frac{\nabla^2}{2m} + \mu]G(\mathbf{r}, t; \mathbf{r}', t') = \delta(\mathbf{r} - \mathbf{r}')\delta(t - t'). \tag{2.7}$$

By Fourier transforming this differential equation we can find the Green's function G_0 in frequency-momentum space:

$$G_0(\varepsilon, \mathbf{p}) = [\varepsilon - \xi_{\mathbf{p}} + i\delta\,\mathrm{sgn}(\xi_{\mathbf{p}})]^{-1}. \tag{2.8}$$

$\xi_p = p^2/2m - \mu$ is the dispersion measured from the chemical potential μ. Formally $\lim_{\delta \to 0}$ is implied.

However this is not the most useful form for more general cases because not all Green's functions can be written as the inverse of a differential operator. In general a single particle Green's function can be written as

$$G(x, x') = -i\langle \mathrm{T}\, \psi_H(x)\psi_H^\dagger(x')\rangle, \tag{2.9}$$

where $x = (\mathbf{r}, t)$. It can be shown that this is the solution to the above differential equation.

The Green's function can be written in terms of the so called retarded and advanced Green's functions. This is useful due to their simpler analytic properties.

$$\Re G(\varepsilon, \mathbf{p}) = \Re G^R(\varepsilon, \mathbf{p}) = \Re G^A(\varepsilon, \mathbf{p}), \tag{2.10}$$

$$\Im G^R(\varepsilon, \mathbf{p}) = \Im G(\varepsilon, \mathbf{p})\,\mathrm{sgn}(\xi_{\mathbf{p}}), \tag{2.11}$$

$$\Im G^A(\varepsilon, \mathbf{p}) = -\Im G(\varepsilon, \mathbf{p})\,\mathrm{sgn}(\xi_{\mathbf{p}}). \tag{2.12}$$

For a single particle with a noninteracting Hamiltonian $\hat{H}_0$, the retarded and advanced Green's function can be written as

$$G^{R/A}(\varepsilon, \mathbf{p}) = (\varepsilon - \xi_p \pm i\delta)^{-1}. \tag{2.13}$$

Lastly, rewriting equation (2.9) in the interaction representation gives

$$G(x, x') = -i \frac{\langle \mathrm{T}\, S(\infty) \psi_I(x) \psi_I^\dagger(x') \rangle}{\langle S(\infty) \rangle}. \tag{2.14}$$

This is the starting point of the zero temperature diagram technique in which $S(\infty)$ is expanded in a Taylor's series.

2.2 The Diagrammatic Technique at Zero Temperature

The Diagram technique has two distinct advantages: one can more easily decide which contributions are small; a set of diagrams can be summed using Dyson's equation which we will introduce later. From equation (2.14) we expand the exponential in the S-matrix, $S(\infty)$:

$$\begin{aligned} G(x, x') = \frac{-i}{\langle S(\infty) \rangle} \sum_{n=0}^{\infty} \frac{(= i)^n}{n!} \prod_{i=1}^{n} \int_{-\infty}^{\infty} dt_i \\ \times \langle \mathrm{T}[\psi_I(x) \psi_I^\dagger(x') \hat{H}_i(t_1) \ldots \hat{H}_i(t_n)] \rangle. \end{aligned} \tag{2.15}$$

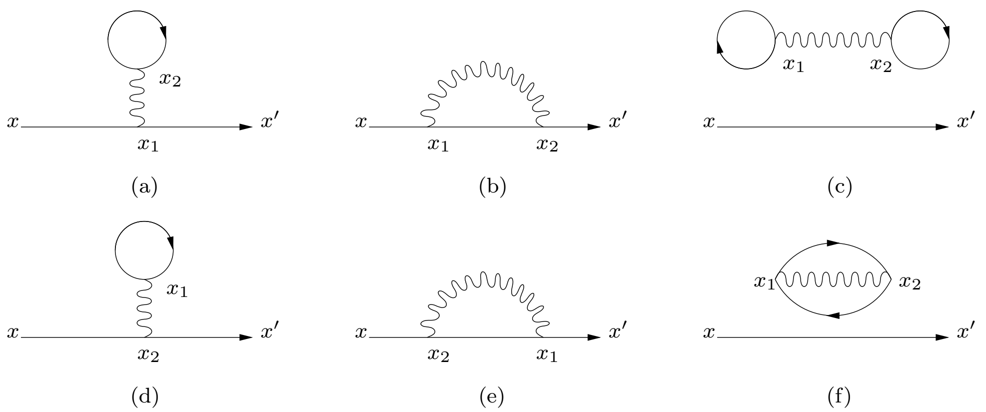
A general (spin independent) two particle interaction can be written a:

$$\hat{H}_i = \frac{1}{2} \int d^4x d^4x' \psi^\dagger(x) \psi^\dagger(x') V(x - x') \psi(x') \psi(x). \tag{2.16}$$

Where V is defined in terms of the interaction U such that $V(x - x') \equiv \delta(t - t') U(\mathbf{r} - \mathbf{r}')$. Now we apply Wick's Theorem[36]. This rewrites the full Green's function as a sum of products of Green's functions and potentials. The first order terms are shown in figure 2.1. As an example, the diagram of figure 2.1(b) is from

$$-\frac{1}{2} \int d^4x_1 d^4x_2 V(x_1 - x_2) G^0(x, x_1) G^0(x_1, x_2) G^0(x_2, x'). \tag{2.17}$$

Simplifications can be made by noticing that the contributions from diagrams (a) and (d) are identical, similarly for (b) and (e). The diagrams (c) and (f) are known as unconnected



(a) (b) (c)

(d) (e) (f)

Figure 2.1: First order terms of expansion of Green's function with a Coulomb potential. Solid lines represent the free particle Green's functions and wavy lines the interaction. All internal indices are integrated over.

diagrams (for obvious reasons), and it can be shown[37] that they do not contribute to the full Green's function. The expansion of $S(\infty)$ in the denominator of equation (2.14) contains only unconnected diagrams and these cancel order by order with those on the numerator.

Now, contributions to an expansion of a full G can be split into two types, reducible and non-reducible. The first are those which it is possible to write in terms of sums of products of simpler contributions, see figure 2.2. Non-reducible diagrams however will contain some kind of "cross term" such that they can not be reduced to a combination of simpler terms, for example see figure 2.2(a). Dyson's equation is a way of writing a sum of reducible diagrams such that

$$\begin{aligned} G(\mathbf{p},\varepsilon) &= G^0 + G^0\Sigma G^0 + G^0\Sigma G^0\Sigma G^0 + \cdots \\ &= [(G^0)^{-1} - \Sigma]^{-1} \to \frac{1}{\varepsilon - \xi_{\mathbf{p}} - i\Im\Sigma(\mathbf{p},\varepsilon)}, \end{aligned} \tag{2.18}$$

which is the sum in figure 2.2(b). The real part of Σ is neglected in this case as $(G^0)^{-1}$ contains μ the chemical potential and the real part can be scaled away into this, from which energies are being measured. Also note that the above is really a symbolic notation and matrix multiplication is implied over all indices and arguments. This reduces to an algebraic expression in the basis in which everything is diagonal. For a translationally invariant system this will be the momentum basis.

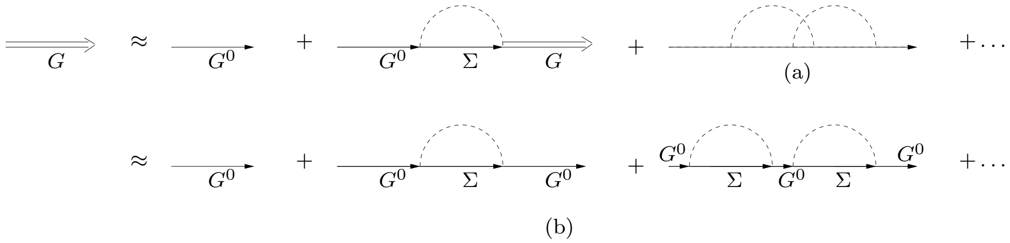


Figure 2.2: An example of a sum of reducible diagrams which can be written in a Dyson's equation form. Diagram (a) is an example of a non-reducible diagram that cannot be included in the sum. However, if we are dealing with disorder averaging for example, as below, then this contribution is less important than those being summed. In this case, as below, the dotted lines refer to the impurity potential contribution $\langle V(\mathbf{r})V(\mathbf{r}')\rangle_i$ of equation (2.20).

2.2.1 Disorder Averaging

The disorder in media is modeled as a Gaussian distribution of impurity potentials, $V(\mathbf{r})$. This is the simplest model for averaging over disorder, which consists of

$$\langle \ldots \rangle_i = \int \frac{DV}{\mathcal{N}_V} \ldots e^{-\pi\nu_d\tau \int V^2(\mathbf{r}) d^d\mathbf{r}} \text{ so} \tag{2.19}$$

$$\langle V(r)\rangle_i = 0 \text{ and } \langle V(r)V(r')\rangle_i = \frac{\delta(r-r')}{2\pi\nu_d\tau}. \tag{2.20}$$

DV is the integral over all realizations of V. Here τ is some phenomenological constant, the scattering rate. $\langle \ldots \rangle_i$ will refer to impurity averaging throughout. The advantage of using Gaussian correlated impurities is that the average depends only on the second moment. Hence, in practise when we wish to average a Green's function or a correlation function we expand the part of S which depends on the impurity potential and then perform disorder averaging. It is then often possible to ignore some higher order contributions and sum the remainder.

The impurity averaged Green's function can be found from the Dyson's equation given above. In this case Σ is the lowest order correction from impurity scattering. It is given by

$$\Sigma(x,x') = \langle V(\mathbf{r})G_0(x-x')V(\mathbf{r}')\rangle_i \tag{2.21}$$

and hence we find

$$\langle G_{dis}(q)\rangle_i = \frac{1}{G_0^{-1}(q) - \Sigma(q)} \tag{2.22}$$

$$= \frac{1}{\omega - \xi_{\mathbf{q}} - i\Im\Sigma(q)} \tag{2.23}$$

$$= \frac{1}{\omega - \xi_{\mathbf{q}} + i\frac{1}{2\tau}\operatorname{sgn}(\omega)} \tag{2.24}$$

as the self energy is

$$i\Im\Sigma(q) = \frac{i}{2\pi\nu\tau}\Im\int\frac{d^d\mathbf{p}}{(2\pi)^d}G_0(q-p)$$

$$= -\frac{i}{2\tau}\operatorname{sgn}(\omega). \tag{2.25}$$

This Green's function is used as the "free electron" propagator in weakly disordered systems and we shall make use of it in chapter 3.

2.2.2 General Results at Zero Temperature for the Diagram Technique

The Keldysh technique of diagrammatics relies on many of the same calculations as at zero temperature. So as an introduction and for later uses various zero temperature results shall be derived here for disordered metals. These diagrams form the "blocks" from which the diagrams for impurity averaged contributions are made. Only lowest order contributions in $\omega\tau$ and pl are considered (l is the mean free path.)

The most basic useful object, beyond a single disorder averaged free particle G_0, is the combination of retarded and advanced Green's functions in a simple "bubble", see figure 2.3(a):

$$K_0(\mathbf{q},\omega) = \int\frac{d^d\mathbf{p}}{(2\pi)^d}G^R(\mathbf{p}+\mathbf{q},\omega)G^A(\mathbf{p},0) = 2\pi\nu_d\tau(1 + i\omega\tau - D\mathbf{q}^2\tau) \tag{2.26}$$

$$\text{using the approximation } \int\frac{d^d\mathbf{p}}{(2\pi)^d} \approx \nu_d\int_{-\varepsilon_F\approx-\infty}^{\infty}d\xi\int\frac{d\Omega}{S_{d-1}}, \tag{2.27}$$

where Ω is the solid angle and S_{d-1} is the requisite surface area. Essentially we are assuming that the density of states is constant near the Fermi surface. This diagram can

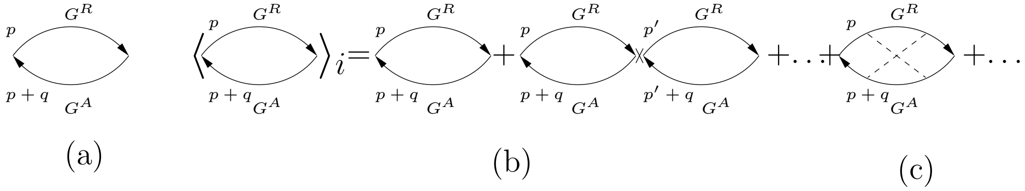


Figure 2.3: A simple $G^{R/A}$ bubble diagram is shown in (a). (b) is the impurity averaged density correlator. We neglect higher order crossing terms such as (c). × denotes impurity scattering.

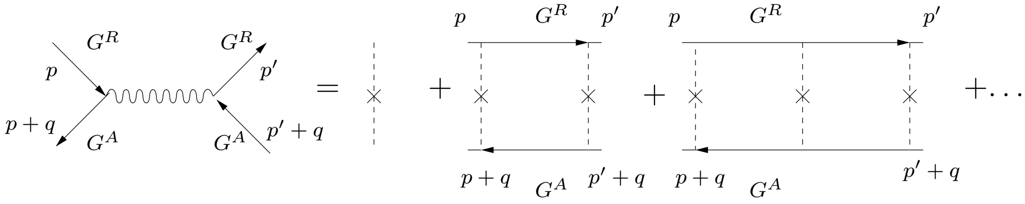


Figure 2.4: A diffuson.

then be used as the basic block in calculating a G^RG^A density correlator and the diffusive modes of a disordered system the diffuson and cooperon.

The density correlator, figure 2.3(b) is

$$\begin{aligned} K(\mathbf{q},\omega) &= \langle G^R(\mathbf{p}+\mathbf{q},\omega)G^A(\mathbf{p},0)\rangle_i \\ &= K_0 + K_0\frac{1}{2\pi\nu_d\tau}K_0 + K_0\frac{1}{2\pi\nu_d\tau}K_0\frac{1}{2\pi\nu_d\tau}K_0 + \cdots \\ &= \frac{2\pi\nu_d}{D\mathbf{q}^2 - i\omega} \end{aligned} \tag{2.28}$$

K_0 is the simple bubble diagram in figure 2.3(a).

The diffuson, as shown in figure 2.4, is found in the same way as K. From the ladder diagram given it is simple to see that

$$D(\mathbf{q},\omega) = \frac{1}{2\pi\nu_d\tau^2(D\mathbf{q}^2 - i\omega)}. \tag{2.29}$$

See appendix B for equivalent diagrams of the diffuson and cooperon and the ladder formulation of the diffuson.

Provided time reversal symmetry holds, the cooperon, given in figure 2.5, is identical

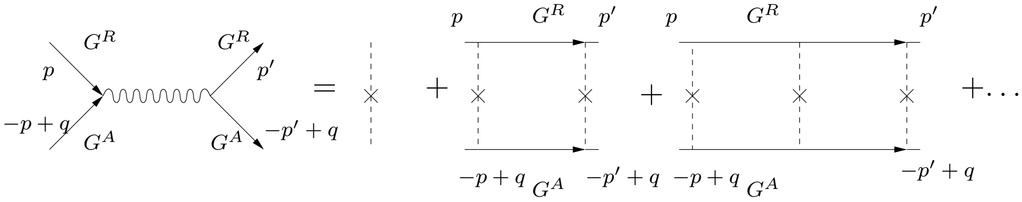


Figure 2.5: A cooperon.

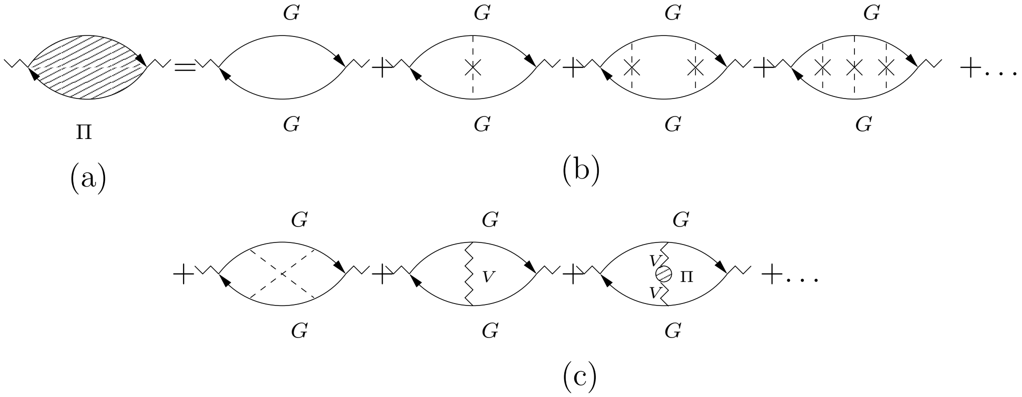


Figure 2.6: A 'full G' polarisation bubble. The last two diagrams in (c) would only arise if we were expanding with an interaction potential also (which we are not here). They are included here as demonstrations of the kind of higher order terms which would be in the screened Coulomb potential expansion. We only include the terms in (b) and neglect all terms like those in (c).

to the diffuson, hence

$$C(\mathbf{q},\omega) = \frac{1}{2\pi\nu_d\tau^2(D\mathbf{q}^2 - i\omega)}. \tag{2.30}$$

The polarisation operator, figure 2.6, is the density correlation function of two Green's Functions in a disordered medium. It is also important for the screened Coulomb propagator as will be seen. The polarisation operator is defined as

$$\begin{aligned}\Pi(\mathbf{q},\omega) &= i\langle G(\mathbf{p}+\mathbf{q},\varepsilon+\omega)G(\mathbf{p},\varepsilon)\rangle_i \\ &= \frac{D\mathbf{q}^2\nu_d}{D\mathbf{q}^2 - i\omega}.\end{aligned} \tag{2.31}$$

For the calculation of $\Pi(q,\omega)$, see appendix C. In figure 2.6 the ladder is given by (b), and the neglected higher order contributions by (c).

Now we also wish to see the effects of disorder on the electron-electron interaction and its vertices. This results in screening of the Coulomb interaction. It is trivial to see that the only event that can happen is given by figure 2.7. Note that while this includes the effects of particle-hole pairs created by the 'photon' scattering from impurities, the effects of the interaction on the particle-hole pair is neglected, as has already been done in calculating the density correlation function equation (2.31). Though, distinctive from high energy physics, this particle-hole pair is created by 'promoting' an electron out of the Fermi sea. Not creating them out of the vacuum, for which the energies are far too small. From the diagram in figure 2.7 can see $V = V_0 + V_0 \Pi V$ which gives

$$V(\mathbf{q}, \omega) = \frac{1}{\frac{1}{V_0(\mathbf{q})} - \Pi(\mathbf{q}, \omega)}. \tag{2.32}$$

The unscreened interaction is given by[13] (where a is the larger transverse length for the quasi 1-d example)

$$V_0(\mathbf{q}) = \begin{cases} e^2 \ln \frac{1}{\mathbf{q}^2 a^2} & \text{if quasi-}d = 1 \\ \frac{2\pi e^2}{\mathbf{q}} & \text{if } d = 2 \\ \frac{4\pi e^2}{\mathbf{q}^2} & \text{if } d = 3. \end{cases} \tag{2.33}$$

The final correction from impurity averaging scattering events is the correction across Coulomb vertices. It is calculated in much the same way as the previous quantities. The Vertex Correction, see figure 2.8, is due to impurity scattering across retarded/advanced Green's functions either side of an emitted or absorbed Coulomb propagator:

$$\Gamma(\mathbf{q}, \omega) = \frac{1}{(D\mathbf{q}^2 - i\omega)\tau}. \tag{2.34}$$

As with the previous quantities, some higher order scattering events have been neglected. Note that this will differ due to whether or not it is emission or absorption and direction of $G^{R/A}$. G^R/Emission/G^A given. To change from emission to absorption we switch $\omega \leftrightarrow -\omega$. To the switch order of G^R and G^A we take the complex conjugate. If however, the process is for example $G^R \rightarrow V \rightarrow G^R$, there is no correction because of the analytic properties of G^R.

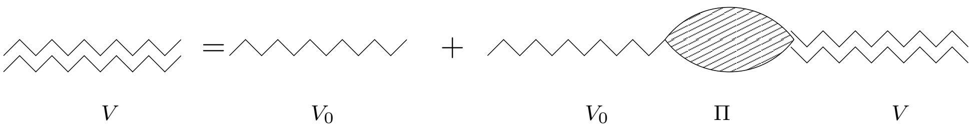


Figure 2.7: The Coulomb interaction screened by impurity scattering.

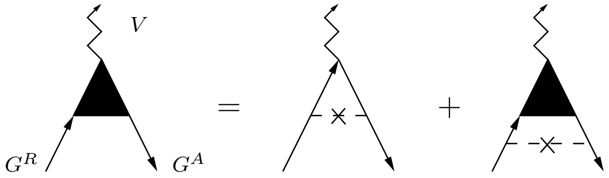


Figure 2.8: The vertex correction across G^R to G^A due to impurity scattering.

2.3 The Keldysh Green's Function

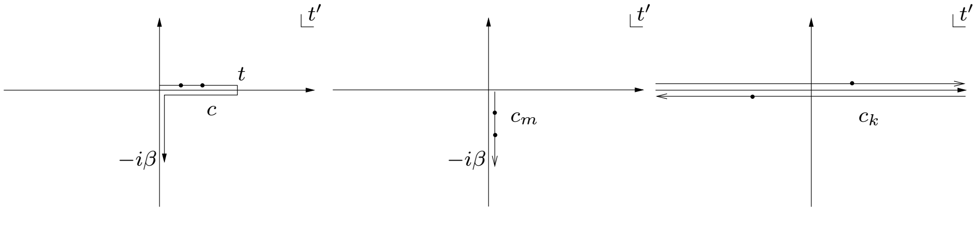
The nonequilibrium form for a Green's function we will present was introduced by Keldysh in 1964[38]. In addition to this work parallel treatments were developed by Martin and Schwinger[39] and Schwinger[40].

Starting from equation (2.9) for finite temperature in the Heisenberg representation we have

$$G(x,x') = -i\langle \mathrm{T}\,\psi_H(x)\psi_H^\dagger(x')\rangle = -i\,\mathrm{Tr}(\rho_0\,\mathrm{T}\,\psi_H(x)\psi_H^\dagger(x')), \tag{2.35}$$

$$\rho_0 = e^{-\beta H} \text{ in equilibrium, the Gibbs distribution.} \tag{2.36}$$

Changing to the interaction representation for ψ and it's conjugate makes the interaction



(a) (b) (c)

Figure 2.9: Interaction contours.

explicit. Note that $e^{-\beta H}$ also requires changing by

$$e^{-\beta H} = e^{-\beta H_0}\,\mathrm{T}\,e^{-\int_0^{-i\beta} dt' H_i(t')}. \tag{2.37}$$

For the Keldysh out of equilibrium formulation the non-interacting part of the Hamiltonian must also contain a term pushing the system out of equilibrium which will in general be time dependent ($\mathcal{H}_0 = H_0 + \mathcal{H}(t)$). This will be assumed rather than explicit in the following calculations. (i.e. H_0 should be replaced by $\mathcal{H}_0$ for Keldysh). We now transform equation (2.9) to the interaction representation. We have an integral over the interaction part of the Hamiltonian with a time contour as in figure 2.9(a).

$$\rightarrow G(x,x') = -\frac{i}{Z}\langle \mathrm{T}_c \underbrace{e^{-i\int_c dt'' H_i(t'')}}_{\equiv S_c} \psi_{H_0}(x)\psi^\dagger_{H_0}(x')\rangle, \tag{2.38}$$

and $Z \equiv \langle S_c\rangle$ is the partition function.

From equation (2.38) it is possible to rotate the real part of the contour c onto the pure imaginary axis. Thus giving the Matsubara contour in figure 2.9(b). This is a single integral from 0 to inverse temperature β in the imaginary time variable defined as $\tau = -it$. Perturbation theory can then be performed and this leads to the Matsubara formalism[37]. Due to the finite length of the imaginary time integral Fourier transforming to frequency space gives sums over so called Matsubara frequencies (for bosonic $\omega = 2n\pi T$ and for fermionic $\omega = (2n+1)\pi T$, chosen for their symmetry and antisymmetry properties).

However we shall use an alternative formalism known as the Keldysh technique. The benefit of using this technique is that we are not required to specify the equilibrium distribution function. Hence it can be used for nonequilibrium problems. We shall see why this is true shortly. (The equilibrium distribution function is specified in the Matsubara technique due to equation (2.37)) For the Keldysh contour the real part of c is extended to $\pm\infty$ and the imaginary part is neglected giving figure 2.9(c). We are free to extend the contour backwards in time as the upper and lower branches will cancel exactly. The imaginary part of the contour describes the initial distribution of the system. At $t = -\infty$ the interactions is switched on adiabatically. The distribution at $t = -\infty$ is not important as we are interested in what happens after the interaction has been turned on. In the equilibrium case it is assumed that the state we are left with after the interactions have been switched on is the new unique ground state[41]. We also assume the state does not depend on the switching mechanism. However, out of equilibrium these assumptions fail. The power of the Keldysh method is that we are not required to specify the state we end up with. This is because we "unevolve" the state again on the lower contour. Consequently we can use this formalism to describe nonequilibrium problems.

The function is now given by

$$G(x, x') = -i\langle \mathrm{T}_{c_k} \, S_{c_k} \psi_{H_0}(x) \psi^{\dagger}_{H_0}(x') \rangle. \tag{2.39}$$

The partition function $Z = \langle S_{c_k} \rangle$ is unity as the upper and lower contours cancel exactly. In order to derive the matrix structure of the Keldysh Green's functions from here, equation (2.39) is split into four separate cases depending on if t and t' are on the upper, lower or separate contours. Labelling the upper contour of figure 2.9(c) as 1 and the lower contour as 2, we write G_{ij}. Then $i = \{1, 2\}$ and $j = \{1, 2\}$ refer to t and t' existing on the upper and lower contour, respectively.

We demonstrated that the single particle Green's function can be written as

$$G(x, x') = -i\langle \mathrm{T}_{c_k} \, S_{c_k} \psi_{H_0}(x) \psi^{\dagger}_{H_0}(x') \rangle. \tag{2.40}$$

Splitting this into its four possible cases, differing as to whether t, t' lie on the upper or lower contours, we can write this as a matrix. With 1 the upper and 2 the lower contour, and i, j referring to t, t' respectively, we have

$$\tilde{\mathbf{G}} = \begin{pmatrix} G_{11} & G_{12} \\ G_{22} & G_{21} \end{pmatrix}. \tag{2.41}$$

Note that any field or interaction introduced into S_{c_k} will also get a matrix structure and tensor vertices will be necessary to show how all of these are to be 'multiplied'. These are described further in section 2.3.1. For G_{12} and G_{21} the time coordinates are always on different parts of the contour (upper or lower). This ensures they are always ordered either forwards (G_{12}) or backwards (G_{21}) along the contour. Hence these terms are equivalent to $G^<$ and $G^>$ respectively[42, 37]. Where $G^{<,>}(t,t')$ means that $t(<,>)t'$. The diagonal elements of $\mathbf{G}$ are time ordered and anti-time ordered (anti-time ordered means that they are ordered backwards along the contour). To summarize, we have

$$\begin{aligned} G_{11}(t,t') &= -i\langle \mathrm{T}\, \psi_{\mathcal{H}_0}(t)\psi^\dagger_{\mathcal{H}_0}(t')\rangle, \\ G_{22}(t,t') &= -i\langle \tilde{\mathrm{T}}\, \psi_{\mathcal{H}_0}(t)\psi^\dagger_{\mathcal{H}_0}(t')\rangle, \\ G_{12}(t,t') &= G^<(t,t') \text{ and } G_{21}(t,t') = G^>(t,t'). \end{aligned} \tag{2.42}$$

$\tilde{\mathrm{T}}$ is the anti-time ordering operator. However, not all of the entries of this matrix are linearly independent and a simpler form can be found by making a rotation in the Keldysh matrix space.

We will make a rotation to the Larkin-Ovchinnikov basis[43]:

$$\mathbf{G} = \mathbf{L}_0 \boldsymbol{\tau}^3 \tilde{\mathbf{G}} \mathbf{L}_0^\dagger. \tag{2.43}$$

$\mathbf{L}_0 = \frac{1}{\sqrt{2}}(\boldsymbol{\tau}^0 - i\boldsymbol{\tau}^2)$ and the $\boldsymbol{\tau}$'s are the Pauli matrices. We will then end up with

$$\mathbf{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}. \tag{2.44}$$

$G^{R/A}$ are the retarded and advanced Green's functions and

$$\begin{aligned} G^R(t,t') &= G_{11}(t,t') - G_{12}(t,t') = G_{21}(t,t') - G_{22}(t,t'), \\ G^A(t,t') &= G_{11}(t,t') - G_{21}(t,t') = G_{12}(t,t') - G_{22}(t,t') \text{ and} \\ G^K(t,t') &= G_{21}(t,t') + G_{12}(t,t') = G_{11}(t,t') + G_{22}(t,t'). \end{aligned} \tag{2.45}$$

Near equilibrium we can write[42]

$$G^K(\mathbf{p},\varepsilon) = h_\varepsilon(G^R(\mathbf{p},\varepsilon) - G^A(\mathbf{p},\varepsilon)). \tag{2.46}$$

It is easy to verify this is exact for equilibrium with h_ε defined as $h_\varepsilon = 1 - 2f_\varepsilon$ (and f_ε is the Fermi distribution). More generally the distribution function must be found from the

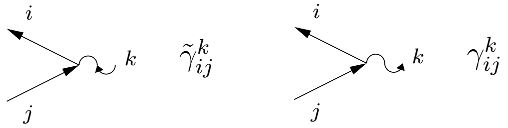


Figure 2.10: Keldysh emission and absorption vertices.

kinetic equation and $G^K(t,t') = G^R(t,t'')\mathcal{F}(t'',t') - \mathcal{F}(t,t'')G^A(t'',t')$. Integration over t'' is implied.

2.3.1 Properties of the Keldysh Matrix Space

The vertices between interactions and the particle Green's functions introduce an asymmetry into the structure. The incoming and outgoing interaction vertices are not the same. This is the price paid for simplifying the matrix structure of $\mathbf{G}$. It is not possible to perform a simple rotation on the Bosonic Green's function to give an upper triangular structure. Hence forcing this to be true has introduced further matrices associated with the Bosonic part which complicate the vertices. Emission and absorption are no longer the same, see figure 2.10. $\tilde{\gamma}^l_{j'i'}$ is the emission vertex and γ^k_{ij} is the absorption vertex given by;

$$\begin{aligned} \gamma^1_{ij} = \tilde{\gamma}^2_{ij} &= \frac{1}{\sqrt{2}}\delta_{ij}, \\ \gamma^2_{ij} = \tilde{\gamma}^1_{ij} &= \frac{1}{\sqrt{2}}\tau^1_{ij}. \end{aligned} \tag{2.47}$$

(δ_{ij} is the normal delta function and τ^1_{ij} is the Pauli matrix).

The rules for constructing the matrix structure of the diagrams, see for example figure 2.11, are relatively straightforward and follow directly from the derivation of the matrix Green's function, equation (2.44). Figure 2.11 is representative of

$$\delta G_{in} = G_{ij}\underbrace{\gamma^o_{jk}(G_{kl}V_{op})\tilde{\gamma}^p_{lm}}_{=\Sigma_{jm}}G_{mn}. \tag{2.48}$$

We have suppressed the time and space dependence. Note that all of the internal Keldysh indices are summed over. Interactions have a matrix structure like equation (2.53) and

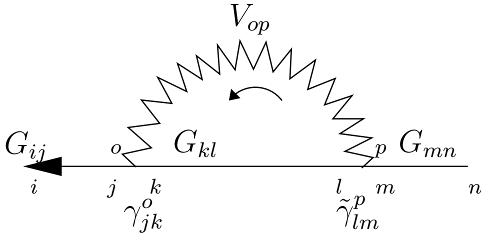


Figure 2.11: An example of the Keldysh structure of a diagram.

the vertices between the interactions and the particles are described by equation (2.47) and figure 2.11. External potentials are diagonal in the Keldysh matrix structure.

2.3.2 Basic Results of the Keldysh Technique

The following are some of the basic "blocks" used when constructing diagrams, which is done analogously to the zero temperature case. The Polarisation Operator, see appendix C, is given by

$$\mathbf{\Pi}(\mathbf{q},\omega) = \begin{pmatrix} \Pi^R(\mathbf{q},\omega) & \Pi^K(\mathbf{q},\omega) \\ 0 & \Pi^A(\mathbf{q},\omega) \end{pmatrix}. \tag{2.49}$$

Where

$$\Pi^R(\mathbf{q},\omega) = [\Pi^A(\mathbf{q},\omega)]^* = \frac{Dq^2\nu_d}{Dq^2 - i\omega}, \tag{2.50}$$

$$\Pi^K(\mathbf{q},\omega) = I(\omega)[\Pi^R(\mathbf{q},\omega) - \Pi^A(\mathbf{q},\omega)] \text{ and} \tag{2.51}$$

$$I(\omega) = \frac{1}{\omega}\int d\varepsilon(1 - h(\varepsilon)h(\varepsilon+\omega)). \tag{2.52}$$

Similarly the screened Coulomb interaction is

$$\mathbf{V}(\mathbf{q},\omega) = \begin{pmatrix} V^R(\mathbf{q},\omega) & V^K(\mathbf{q},\omega) \\ 0 & V^A(\mathbf{q},\omega) \end{pmatrix}, \tag{2.53}$$

with

$$V^R(\mathbf{q},\omega) = [V^A(\mathbf{q},\omega)]^* = \frac{1}{\frac{1}{V_0} - \Pi^R(\mathbf{q},\omega)} \quad \text{and} \tag{2.54}$$

$$V^K(\mathbf{q},\omega) = I(\omega)[V^R(\mathbf{q},\omega) - V^A(\mathbf{q},\omega)]. \tag{2.55}$$

$I(\omega)$ is defined as in equation (2.52). These results are used in the same way as at zero temperature with the added complication of the matrix structure. For examples see appendices C and D.

2.4 Functional Integrals

A functional integral is a path integral defined with the overcomplete set of coherent states. We need to introduce the idea of a coherent state and, for Fermionic functional integrals, of Grassmann algebra[36].

2.4.1 Coherent States and Grassmann Algebra

A coherent state is defined as the eigenstate of an annihilation operator. Note that a creation operator has no eigenstate. Or, strictly speaking, no *right* eigenstate. For bosons, an eigenstate satisfies

$$\hat{a}_\alpha|\phi\rangle = \phi_\alpha|\phi\rangle. \tag{2.56}$$

We can write

$$|\phi\rangle = e^{\sum_\alpha \phi_\alpha \hat{a}^\dagger_\alpha}|0\rangle. \tag{2.57}$$

For bosons the eigenvalues are always complex numbers. However, in the case of fermions these eigenvalues must anticommute to preserve the properties of the fermion annihilation operators.

For the purpose of defining fermionic coherent states we introduce an anticommuting algebra of Grassmann numbers. This is a set of numbers $\{\psi_\alpha\}$ which obey the anticommutation relation

$$\{\psi_\alpha, \psi_\beta\} = 0. \tag{2.58}$$

This means that a Grassmann number has the property that it's square is zero and hence any function of a Grassmann number must be linear.

We can define the conjugate of a product of Grassmann numbers as

$$(\psi_{\alpha_1}\psi_{\alpha_2}\ldots\psi_{\alpha_n})^* = \psi^*_{\alpha_n}\ldots\psi^*_{\alpha_2}\psi^*_{\alpha_1}. \tag{2.59}$$

We also require $(\psi^*)^* = \psi$. We can define integration and differentiation:

$$\frac{d}{d\psi}\psi = 1 \quad \int d\psi 1 = 0 \quad \int d\psi\psi = 1. \tag{2.60}$$

Note that for a derivative to act it must be anticommuted through all other fields so that it is adjacent to the field it is acting on.

Using this algebra we can now define a fermionic coherent state:

$$\hat{a}_\alpha|\psi\rangle = \psi_\alpha|\psi\rangle \qquad |\psi\rangle = \prod_\alpha(1 - \psi_\alpha\hat{a}^\dagger_\alpha)|0\rangle. \tag{2.61}$$

One property of these states we shall require is their overcompleteness[36]

$$\int\prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\alpha\bar{\psi}_\alpha\psi_\alpha}|\psi\rangle\langle\psi| = 1. \tag{2.62}$$

A similar relation can be found for the bosonic case. We shall also use the property, for a general operator $\hat{A}$, that

$$\mathrm{Tr}\,\hat{A} = \int D\psi D\bar{\psi}e^{\bar{\psi}\psi}\langle-\psi|\hat{A}|\psi\rangle \tag{2.63}$$

to derive the Green's function. We have defined $D\psi \equiv \prod_\alpha d\psi_\alpha$ and $\bar{\psi}\psi \equiv \sum_\alpha\bar{\psi}_\alpha\psi_\alpha$.

2.4.2 The Functional Integral Green's Function

We shall derive the expression for a Green's function in the functional integral representation. Starting from

$$\begin{aligned} G(t,t') &= -\frac{i}{Z}\sum_n \langle n|T_c\hat{\psi}_t\psi^\dagger_{t'}e^{-i\int_c dt_1 H(t_1)}|n\rangle \\ &= -\frac{i}{Z}\int D\psi_0 D\bar{\psi}_0 e^{\bar{\psi}_0\psi_0}\langle -\psi_0|T_c\hat{\psi}_t\hat{\psi}^\dagger_{t'}e^{-i\sum_i \delta_i H(\hat{\psi}^\dagger_i,\hat{\psi}_i)}|\psi_0\rangle \end{aligned} \tag{2.64}$$

we can define $\psi_0 = -\psi_{N+1}$ and split the time contour into N pieces of width δ_i. We now insert N completeness relations at these points:

$$\begin{aligned} G(t,t') = -\frac{i}{Z}\int & D^{N+1}\psi D^{N+1}\bar{\psi}e^{\bar{\psi}_0\psi_0-\sum_{i=1}^{N}\bar{\psi}_i\psi_i}\langle\psi_{N+1}|e^{-i\delta_N H[\hat{\psi}^\dagger,\hat{\psi}]}|\psi_N\rangle\langle\psi_N| \\ & \ldots e^{-i\delta_1 H[\hat{\psi}^\dagger,\hat{\psi}]}|\psi_1\rangle\langle\psi_1|e^{-i\delta_0 H[\hat{\psi}^\dagger,\hat{\psi}]}\hat{\psi}_t\hat{\psi}^\dagger_{t'}|\psi_0\rangle. \end{aligned} \tag{2.65}$$

Here the 0 and i subscripts refer to the different completeness relations inserted, not the analogue of the label α used in the preceding section. This can be simplified using the properties of the coherent states to give

$$\begin{aligned} G(t,t') &= -\frac{i}{Z}\int \underbrace{D^{N+1}\psi D^{N+1}\bar{\psi}}_{\to D\psi D\bar{\psi}} e^{iS}\psi_t\bar{\psi}_{t'} \text{ where} \\ iS &= \bar{\psi}_0\psi - \sum_{i=1}^{N}\bar{\psi}_i\psi_i + \sum_{i=0}^{N}\{\bar{\psi}_{i+1}\psi_i - i\delta_i H[\bar{\psi}_{i+1},\psi_i]\} \\ &= \sum_{i=0}^{N}\delta_i\left\{\bar{\psi}_{i+1}\frac{\psi_i-\psi_{i+1}}{\delta_i} - iH[\bar{\psi}_{i+1},\psi_i]\right\} \\ &\to i\int_c dt\left\{\bar{\psi}(t)i\frac{d\psi(t)}{dt} - H[\bar{\psi}(t),\psi(t)]\right\}. \end{aligned} \tag{2.66}$$

It should be noted that the continuum expression is strictly symbolic and that the discrete expression for the action S is the only one with any real meaning.

The partition function is calculated in the same way, yielding

$$Z = \int D\bar{\psi}D\psi e^{iS}. \tag{2.67}$$

It is trivial to generalise this to the case where there is more than one state for the particle to carry (and hence there are additional labels on the operators and states.) The states and operators simply carry extra labels.

2.4.3 Some Basic Results

Consider the matrix $\mathbf{M}$ defined by

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 & \dots & a_{N+1} \\ -a_1 & 1 & 0 & \dots & 0 \\ 0 & -a_2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & -a_N & 1 \end{pmatrix}. \tag{2.68}$$

The entries are defined by $a_i = 1 - i\delta_i\phi_i$. i labels the times and ϕ_i is the Hamiltonian at time t_i, compare with equation (2.66). Then

$$-\int D\psi D\bar{\psi} e^{-\bar{\psi}\mathbf{M}\psi} = -\det\mathbf{M} = 1 + \prod_{i=1}^{N+1} a_i \text{ and} \tag{2.69}$$

$$\int D\psi D\bar{\psi} e^{-\bar{\psi}\mathbf{M}\psi}\psi_k\bar{\psi}_l = \begin{cases} -\prod_{i=l}^{k+1} a_i & \text{if } k > l \\ \prod_{i=1}^{k-1}\prod_{l}^{N+1} a_i & \text{if } k < l. \end{cases} \tag{2.70}$$

Hence, for $a_i = 1 - i\delta_i\varphi_i \approx e^{-i\delta_i\varphi_i}$, where δ_i is a time segment, we can write

$$\int D\psi D\bar{\psi} e^{-\bar{\psi}\mathbf{M}\psi} = 1 + e^{-i\int\varphi(t)dt} \text{ and} \tag{2.71}$$

$$\int D\psi D\bar{\psi}\psi(t)\bar{\psi}(t')e^{-\bar{\psi}\mathbf{M}\psi} = \begin{cases} e^{-i\int_{t'}^{t}\varphi(t'')dt''} & \text{if } t > t' \\ -e^{-i\int_{t_0}^{t}\varphi(t'')dt''-i\int_{t'}^{t_{N+1}}\varphi(t'')dt''} & \text{if } t < t'. \end{cases} \tag{2.72}$$

If we can not write such a simple form for the matrix $\mathbf{M}$ we will use the standard result:

$$\int D\psi D\bar{\psi} e^{-\bar{\psi}_\alpha\mathbf{M}_{\alpha\beta}\psi_\beta} = \det\mathbf{M}. \tag{2.73}$$

$$\tag{2.74}$$

The labels α and β refer to any additional structure: for example spin or momentum.

Chapter 3

THE ZERO BIAS ANOMALY

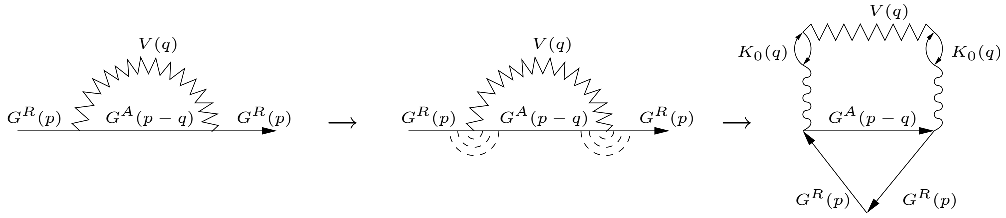
3.1 Perturbation Theory

In this chapter we will present a calculation of the tunnelling density of states for the zero bias anomaly. This is a generalisation, to the Keldysh method, of previous work by Kamenev and Gefen[24]. We find the correction to the one particle tunnelling density of states due to the self energy of the electron-electron interaction and it's screening from impurities. This includes the effects of disorder on the vertices and screening of the Coulomb interaction. As we shall see the zero momentum mode of the interaction cannot be taken into account in this method for small temperatures. When it is the most important contribution we shall need to do something beyond perturbation theory. Dealing with this mode beyond perturbation theory is the task of the following chapter. Here we shall repeat the calculation of Kamenev and Gefen (though in the Keldysh formulation) and compare it to the classic Altshuler-Aronov result[13]. This then leads us to consider the zero momentum mode of the interaction correctly in the next chapter, where we believe that Kamenev and Gefen have not correctly accounted for it.

The density of states can be written as

$$\nu_d(\varepsilon) = -\frac{1}{\pi}\Im \int \frac{d^d\mathbf{p}}{(2\pi)^d} G^R(\mathbf{p},\varepsilon). \tag{3.1}$$

For a free particle Green's function, G^0, equation (3.1) just returns the usual density of states for each dimensionality. Expanding G in the interaction strength will give the



(a) (b) (c)

Figure 3.1: Diagrams for the Fock contribution to the density of states correction. Figures (b) and (c) are after disorder averaging. See appendix B for alternative diffuson diagrams.

corrections to the density of states. The first order correction will be

$$\delta\nu_d^1(\varepsilon) = -\frac{1}{\pi}\Im\int \frac{d^d\mathbf{p}}{(2\pi)^d}[G^R]^1(\mathbf{p},\varepsilon). \tag{3.2}$$

The label ^{1} signifies that it is the first order correction. We shall also average over the disorder potential and sum all lowest order diagrams this produces.

Starting from a single particle Green's function with a Coulomb interaction V, the interaction Hamiltonian has the form[37]

$$H_{int} = \frac{1}{2}\int d^{d+1}x_1 d^{d+1}x_2 \psi^\dagger(x_1)\psi^\dagger(x_2)V(x_1-x_2)\psi(x_2)\psi(x_1). \tag{3.3}$$

We expand the S-matrix in the interaction representation and keep only the first order correction in V, see figure 3.1(a). Note that this is before impurity averaging. After impurity averaging, the vertices become "dressed" and we have figure 3.1(b). On performing the momentum integral over $\mathbf{p}$ in equation(3.16) this will then be represented by figure 3.1(c). Note that the triangle in (c) is only closed in momentum space, not in frequency space.

In the interaction representation the Green's function is, see section 2.1.1,

$$G(x,x') = -i\frac{\langle \mathrm{T}\, S(\infty)\psi_I(x)\psi_I^\dagger(x')\rangle}{\langle S(\infty)\rangle}. \tag{3.4}$$

The S-matrix, which we will Taylor expand, is

$$S(\infty) = \mathrm{T}\, e^{-i\int_{-\infty}^{\infty} H_{int}(t')dt'}. \tag{3.5}$$

After Taylor expanding the S-matrix and applying Wick's theorem[36, 37] we can extract the first order correction. (First order in the interaction.) There are four contributions at first order. Two involve the cooperon corrections which we do not consider[13]. The remaining two are the diffuson corrections. These are referred to as the Fock, or exchange, correction and the Hartree correction. We will teat the Hartree correction in section 3.1.3. It transpires that we are able to neglect this. Here we present only the Fock term. The Fock correction is the term[13]

$$\delta G^1 = i \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{d\omega}{2\pi} G(P)G(P-Q)V(Q)G(P). \tag{3.6}$$

Here $G(P)$ are unaveraged with respect to the disorder potential, and P and Q are 4-vectors. Note that V will differ depending on the dimension. The diagram for equation (3.6) is figure 3.1(a).

In the Keldysh technique[42], see section 2.3, this will become, for the retarded Green's function,

$$\delta G^1_{11}(P) = i \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{d\omega}{2\pi} G_{1i}(P)G_{jj'}(P-Q)G_{i'1}(P)\gamma^k_{ij}V_{kl}(Q)\tilde{\gamma}^l_{j'i'}. \tag{3.7}$$

Now, using the analytic properties of the above functions and their Keldysh structure, namely $G_{21} = V_{21} = 0$, equation (3.7) can be written as

$$\begin{aligned} \delta G^1_{11}(P) = \frac{i}{2} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{d\omega}{2\pi} &[G_{11}(P)G_{12}(P-Q)G_{11}(P)V_{11}(Q) \\ &+ G_{12}(P)G_{22}(P-Q)G_{11}(P)V_{11}(Q)]. \end{aligned} \tag{3.8}$$

The analytic properties of the retarded and advanced Green's functions ensures that, upon integration, some of the terms in equation (3.7) are zero. So equation (3.2) becomes

$$\begin{aligned} \delta\nu^1_d(\varepsilon) = -\frac{1}{2\pi}\Im i \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{d\omega}{2\pi} V_{11}(Q)[G_{11}(P)G_{12}(P-Q)G_{11}(P) + \\ G_{12}(P)G_{22}(P-Q)G_{11}(P)]. \end{aligned} \tag{3.9}$$

Averaging over the disorder potential cannot alter the Keldysh matrix structure and the result of this process is simple to see from diagrammatics, see figure 3.1. The first term in equation (3.9) will reduce to, using analytic properties and $G_{12}(Q) = h(\omega)(G_{11}(Q) - G_{22}(Q))$,

$$-h(\varepsilon - \omega)G_{11}(P)G_{22}(P-Q)G_{11}(P)V_{11}(Q). \tag{3.10}$$

Performing disorder averaging yields

$$-h(\varepsilon - \omega)G_{11}(P)G_{22}(P-Q)G_{11}(P)V_{11}(Q)\frac{(2\pi\nu_d\tau)^2}{[2\pi\nu_d\tau^2(D\mathbf{q}^2 - i\omega)]^2} \tag{3.11}$$

$$\rightarrow \frac{-h(\varepsilon - \omega)G_{11}(P)G_{22}(P-Q)G_{11}(P)V_{11}(Q)}{\tau^2(D\mathbf{q}^2 - i\omega)^2}. \tag{3.12}$$

This includes the diffuson propagators in figure 3.1. Similarly, the second term in equation (3.9) will give the following after disorder averaging:

$$\frac{h(\varepsilon)G_{11}(P)G_{22}(P-Q)G_{11}(P)V_{11}(Q)}{\tau^2(D\mathbf{q}^2 - i\omega)^2}. \tag{3.13}$$

This ignores higher order terms from crossed impurity lines.

Now, to lowest order the integral over $\mathbf{p}$ is trivial to perform and leads to

$$\int \frac{d^d\mathbf{p}}{(2\pi)^d} G_{11}(P)G_{22}(P-Q)G_{11}(P) \approx 2\pi i\nu_d\tau^2. \tag{3.14}$$

Note that to lowest order $G(P-Q) \approx G(P)$ and so the integral over $\mathbf{p}$ can be performed without needing to worry about the convergence of the ω integral. Collating the above results, the density of states correction becomes

$$\frac{\delta\nu_d^1(\varepsilon)}{\nu_d} = \Im \int_{L^{-1}<|\mathbf{q}|<l^{-1}} \frac{d^d\mathbf{q}}{(2\pi)^d} \int_{-\frac{1}{\tau}}^{\frac{1}{\tau}} \frac{d\omega}{2\pi} \frac{V_{11}(\mathbf{q},\omega)}{(D\mathbf{q}^2 - i\omega)^2}[h(\varepsilon) - h(\varepsilon - \omega)]. \tag{3.15}$$

This can be rewritten as

$$\frac{\delta\nu_d^1(\varepsilon)}{\nu_d} = \Im \int_{L^{-1}<|\mathbf{q}|<l^{-1}} \frac{d^d\mathbf{q}}{(2\pi)^d} \int_{0}^{\frac{1}{\tau}} \frac{d\omega}{2\pi} \frac{V_{11}(\mathbf{q},\omega)}{(D\mathbf{q}^2 - i\omega)^2}[h(\varepsilon + \omega) - h(\varepsilon - \omega)]. \tag{3.16}$$

The limits on the integrals are due to the conditions $\omega\tau \ll 1$ and $|\mathbf{q}|l \ll 1$ with $|\mathbf{q}| > L^{-1}$ (L is the system size). We are interested in the regime described by these approximations.

3.1.1 Two Dimensions

As there is no angular dependence on **q** in the integrand we can write

$$\int \frac{d^2\mathbf{q}}{4\pi^2} \to \frac{1}{2\pi}\int_{L^{-1}}^{l^{-1}} dqq. \tag{3.17}$$

Furthermore

$$V^R(\mathbf{q},\omega) = \frac{2\pi e^2}{q - \frac{D\kappa_2 \mathbf{q}^2}{D\mathbf{q}^2 - i\omega}}, \tag{3.18}$$

$$\text{where } \kappa_2 = 2\pi e^2 \nu_2. \tag{3.19}$$

The most important contribution comes from the region of integration: $|\mathbf{q}| \ll \sqrt{\frac{\omega}{D}}$. Calculating and rearranging the above case yields

$$\delta\nu_2^{(1)}(\varepsilon) = \frac{1}{8\pi^2 D}\int_0^{\frac{1}{\tau}} \frac{d\omega}{\omega} \ln\frac{\omega}{D\kappa_2^2}[h(\varepsilon+\omega) - h(\varepsilon-\omega)]. \tag{3.20}$$

For the equilibrium case this gives the following usual result[13]:

$$\delta\nu_2^{(1)}(\varepsilon) = \frac{1}{8\pi^2 D}\int_0^{\frac{1}{\tau}} \frac{d\omega}{\omega} \ln\frac{\omega}{D\kappa_2^2}\Big[\tanh\Big(\frac{\omega+\varepsilon}{2T}\Big) + \tanh\Big(\frac{\omega-\varepsilon}{2T}\Big)\Big]. \tag{3.21}$$

So for very low temperatures, $T \ll (\omega+\varepsilon)$, there is a cut-off at $\omega = \varepsilon$ instead of τ^{-1}.

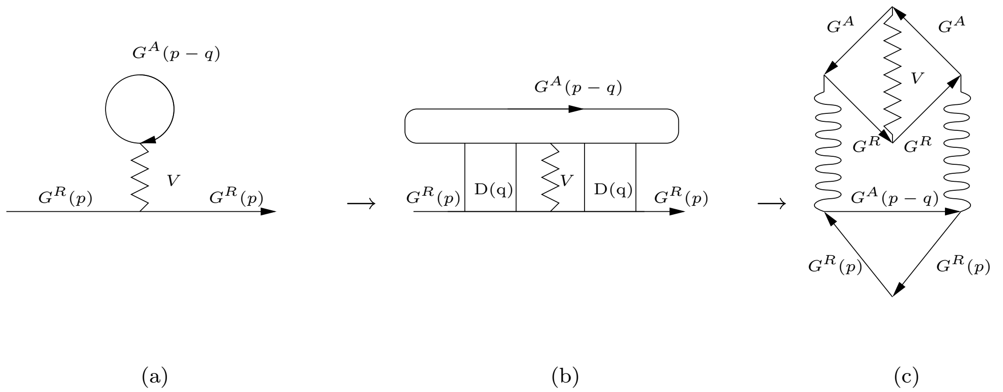
3.1.2 Three Dimensions

For three dimensions the screened Coulomb potential looks like

$$V^R(\mathbf{q},\omega) = \frac{4\pi e^2}{\mathbf{q}^2 - \frac{D\kappa_3^2 \mathbf{q}^2}{D\mathbf{q}^2 - i\omega}}, \tag{3.22}$$

$$\text{where } \kappa_3^2 = 4\pi e^2 \nu_3. \tag{3.23}$$

The integral over frequencies now appears to be divergent. However it is not a physical divergence and can be removed by subtracting a constant from the density of states.



(a) (b) (c)

Figure 3.2: Diagrams for the Hartree density of states correction. Figures (b) and (c) are after disorder averaging. See appendix B for alternative diffuson diagrams.

Subtracting a constant to remove the divergence will leave

$$\frac{\delta\nu_3^1(\varepsilon)}{\nu_3} = \Im \int_{L^{-1}<|\mathbf{q}|<l^{-1}} \frac{d^3\mathbf{q}}{(2\pi)^3} \int_0^{\frac{1}{\tau}} \frac{d\omega}{2\pi} [h(\varepsilon+\omega) - h(\varepsilon-\omega) - 2] \frac{V(\mathbf{q},\omega)}{D\mathbf{q}^2 - i\omega}. \tag{3.24}$$

We are interested in the zero dimensional limit of this quantity. This is calculated in the next section.

3.1.3 The Hartree Term

Now we also wish to calculate the Hartree correction which should be small for long range potentials such as the Coulomb potential. This is to demonstrate we are justified in ignoring it. The Hartree correction looks like figure 3.2. It is calculated in a similar way to the previous Fock contribution. For simplicity the two dimensional case will be done for the screened Coulomb potential. If a short range potential is specified the answer will be of a similar order to the Fock case and we are no longer justified in ignoring it.

The Hartree correction from figure 3.2(c) is

$$\begin{aligned} \frac{\delta\nu_2^1(\varepsilon)}{\nu_2} &= \Im \int_{L^{-1}<|\mathbf{q}|<l^{-1}} \frac{d^2\mathbf{q}}{(2\pi)^3} \int_0^{\frac{1}{\tau}} \frac{d\omega}{2\pi} \frac{2V\tau[h(\varepsilon-\omega) - h(\varepsilon+\omega)]}{\pi\nu_2(D\mathbf{q}^2 - i\omega)^2} \\ &\approx \frac{V\tau}{4\pi^3 D\nu_2} \int_0^\infty \frac{d\omega}{\omega}[h(\varepsilon+\omega) - h(\varepsilon-\omega)]. \end{aligned} \tag{3.25}$$

For zero temperature, the correction is of order $\ln(\varepsilon\tau)$ compared with $(\ln(\varepsilon\tau))^2$ for the Fock term and can therefore clearly be safely ignored.

3.2 The Tunnelling Density of States

From the three dimensional density of states we will take the $L \ll l$ limit. Hence integration over the momenta is no longer valid and we use the correct quantized form for the momentum:

$$\int \frac{d^d\mathbf{q}}{(2\pi)^d} \rightarrow \frac{1}{L^d}\sum_{\mathbf{q}} \tag{3.26}$$

$$\text{with } \mathbf{q} = \frac{2\pi}{L}\mathbf{n}. \tag{3.27}$$

Here L^d is the size of the quasi-zero dimensional object, which we shall assume to be square, and $\mathbf{n}$ is a d dimensional vector whose entries are integers.

We need to analyse the term from equation (3.24):

$$\Im\frac{V(\mathbf{q},\omega)}{D\mathbf{q}^2 - i\omega} = \frac{4\pi e^2}{\mathbf{q}^2}\frac{\omega(2D\mathbf{q}^2 - D\kappa_3^2)}{[D\mathbf{q}^2(D\mathbf{q}^2 - D\kappa_3^2) - \omega^2]^2 + \omega^2(2D\mathbf{q}^2 - D\kappa_3^2)^2}. \tag{3.28}$$

For the $d \rightarrow 0$ case we can use $D\mathbf{q}^2 \approx D|\mathbf{n}|/L^2 = E_{Th}n \gg \omega$, as $q \approx L^{-1}$. E_{Th} is the definition Thouless energy. Obviously we must treat $n \neq 0$ separately. Making this approximation and focusing on the most divergent term, $\mathbf{q} \gg \kappa_3$, we arrive at

$$\frac{\delta\nu^1(\varepsilon)}{\nu} = -\frac{1}{L^3}\int_0^{\frac{1}{\tau}}\frac{d\omega}{2\pi}\omega[h(\varepsilon+\omega) - h(\varepsilon-\omega) - 2]\sum_{\mathbf{q}\neq 0}\frac{4\pi e^2}{D^3\kappa_3^2}\frac{1}{\mathbf{q}^6}. \tag{3.29}$$

In terms of the dimensionless conductance, $g = \nu_d D L^{d-2}$, and with $a_0 = \sum_{\mathbf{n}\neq 0}(2\pi|\mathbf{n}|)^{-6}$ we obtain

$$\frac{\delta\nu^1(\varepsilon)}{\nu} = a_0\frac{1}{g}\left(\frac{T}{E_c}\right)^2 \mathrm{f}\left(\frac{\varepsilon}{T}\right), \tag{3.30}$$

$$\text{where } \mathrm{f}(x) = \int_0^\infty \frac{dy}{2\pi}y\big(2 - h[(x+y)T] + h[(x-y)T]\big) \tag{3.31}$$

is a dimensionless integral. Substituting the equilibrium distribution function for $h(\varepsilon)$ is then in agreement with the Kamenev and Gefen work. However as we are in a pseudo-zero dimensional regime we would expect the zero mode to be the most important. Incorporating this is not trivial however as it is unscreened and the bare Coulomb interaction is ill defined for $\mathbf{q} = 0$.

The method proposed to circumnavigate this problem was to introduce by hand a 'regularising' mechanism for the $\mathbf{n} = 0$ mode[24]. At $\mathbf{q} \approx L^{-1}$ the sample will be regularised and with a self capacitance $C \approx L$ we use

$$E_c \sim \frac{e^2}{C} \tag{3.32}$$

as the zero mode interaction. An inelastic relaxation rate, γ_{in}, is also introduced into the diffusons in the Fock term of the density of states correction. Strictly this should always be there as it ensures the Diffusons have a finite lifetime but it is normally not important. So for the zero mode contribution we have

$$\begin{aligned} \left.\frac{\delta\nu^1(\varepsilon)}{\nu}\right|_{\mathbf{n}=0} &= \int_0^\infty \frac{d\omega}{2\pi}[h(\varepsilon+\omega) - h(\varepsilon-\omega) - 2]\Im\frac{e^2}{C}\frac{1}{(\gamma_{in} - i\omega)^2} && (3.33) \\ &= \frac{e^2}{C}2\gamma_{in}\int_0^\infty \frac{d\omega}{2\pi}\frac{\omega}{(\gamma_{in}^2+\omega^2)^2}[h(\varepsilon+\omega) - h(\varepsilon-\omega) - 2]. && (3.34) \end{aligned}$$

At equilibrium this equation can be found in terms of digamma functions and calculated in the limit of $\gamma_{in} \ll T$[24]:

$$\left.\frac{\delta\nu^1(\varepsilon)}{\nu}\right|_{\mathbf{n}=0} = -\frac{E_c}{4T}\,\mathrm{sech}^2\left[\frac{\varepsilon}{2T}\right]. \tag{3.35}$$

This agrees with the Altshuler Aronov result[13] at $d = 0$[24]. Temperatures below E_c cause problems however.

3.3 Beyond Perturbation Theory

The zero mode contribution will lead to a singularity in the first order correction to the tunnelling density of states[24]. So for small enough temperatures, perturbation theory will no longer be adequate. The zero mode needs to be treated more carefully than in the preceding calculation. Kamenev and Gefen consider a Green's function with a Hamiltonian which contains only the zero mode interaction, i.e. the Coulomb blockade interaction. This is dealt with in the next chapter.

Chapter 4

THE DENSITY OF STATES AND CONDUCTANCE OF A QUANTUM DOT

In this chapter we derive an expression for the Green's function of an isolated quantum dot with the universal Hamiltonian in equation (1.8). This is used to find the average particle number and the tunnelling density of states and their behaviour with changing chemical potential. We use the Keldysh formalism to calculate these properties. The analogous component in the Keldysh formalism to the zero frequency bosonic field in the Matsubara technique can be dealt with in two ways. (This is the component which is most difficult to treat and requires some care, see section 1.5.) We present them both as they allow easier interpretation and comparison with previous work. From the tunnelling density of states it is possible to find the differential conductance through the dot coupled to two leads (reservoirs) which is then compared with the standard result. We find that this can only be correctly reproduced with our corrections to the tunnelling density of states.

4.1 Functional Integral Form for the Green's Function

We want an expression for the single particle Green's function of an electron. The Hamiltonian describing the interacting electrons is, in the grand canonical ensemble,

$$\begin{aligned}\hat{H} - \mu\hat{N} &= \hat{H}_0 - \mu\hat{N} + \frac{E_c}{2}(\hat{N} - N_0)^2 \\ &= \hat{H}_0 - (\mu + E_c N_0)\hat{N} + \frac{E_c}{2}\hat{N}^2 + \text{constant}.\end{aligned} \tag{4.1}$$

The noninteracting part of the Hamiltonian is defined as

$$\hat{H}_0 = \int d\mathbf{r}\hat{\psi}^\dagger(\mathbf{r})\hat{\varepsilon}\hat{\psi}(\mathbf{r}). \tag{4.2}$$

N_0 is the contribution from the background charge of the dot, which is very large and scales with the volume. We consider the electrons to be in a random potential and transform to the basis where

$$\hat{H}_0 = \sum_n \hat{\psi}_n^\dagger \varepsilon_n \hat{\psi}_n. \tag{4.3}$$

n labels the random distribution of energy levels in the dot. We consider the mean level spacing of the energy levels to be less than all other relevant energy scales. In a zero dimensional system all physical properties can then be found in terms of $iG_k(t,t')$.

We can derive a functional integral expression for the Green's function defined on the full interaction contour. We take as our starting point the following expression,

$$iG_n(t,t') = \mathcal{Z}^{-1}\sum_m \langle m|\,\mathrm{T}_c\,\hat{\psi}_n(t)\hat{\psi}_n^\dagger(t')U_c|m\rangle. \tag{4.4}$$

We wish to write this as a functional integral. In the usual way a functional integral form with a general Hamiltonian can be written as

$$iG_n(t,t') = \mathcal{Z}^{-1}\int D\psi D\bar{\psi}\psi_n(t)\bar{\psi}_n(t')e^{i\int_c dt(\sum_k \bar{\psi}_k(t)i\partial_t\psi_k(t) - H)}, \tag{4.5}$$

with the Hamiltonian, H, defined in equation (4.1).

If we perform the Hubbard-Stratonovich transformation on the Green's function, see appendix E, we find

$$iG_n(t,t') = \frac{\int D\phi e^{iS[\phi]} \int D\psi D\bar{\psi}\psi_n(t)\bar{\psi}_n(t') e^{iS[\bar{\psi},\psi,\phi]}}{\int D\phi e^{iS[\phi]} \int D\psi e^{iS[\bar{\psi},\psi,\phi]}} \text{ where} \tag{4.6}$$

$$iS[\bar{\psi},\psi,\phi] = i\sum_k \int_c dt \bar{\psi}_k(t)[i\partial_t - \xi_k - i\phi(t)]\psi_k(t) \text{ and} \tag{4.7}$$

$$iS[\phi] = -i\frac{1}{2E_c}\int_c dt\phi^2(t) - N_0 \int_c dt\phi(t) \tag{4.8}$$

are the new actions. In order to correctly derive this form it is necessary to include terms of the order $(\phi_i\delta_i)^2$, where δ_i is a time segment, and to ensure the interaction is in the normal ordered form. This is reviewed in appendix E.

After the Gaussian Fermionic integrals have been calculated we have an expression for the Green's function in terms of the bosonic field:

$$iG_n(t,t') = \mathrm{sgn}_c(t,t')\frac{\int D\phi e^{iS[\phi]} e^{\int_{c_{tt'}} dt(\phi(t)-i\xi_n)} \prod_{k\neq n}[1+e^{\int_c dt(\phi(t)-i\xi_k)}]}{\int D\phi e^{iS[\phi]} \prod_k [1+e^{\int_c dt(\phi(t)-i\xi_k)}]}. \tag{4.9}$$

$\mathrm{sgn}_c(t,t')$ is defined upon the interaction contour c. The interaction contour is given by figure 2.9(a). We have also defined the following contour:

$$\int_{c_{tt'}} dt = \begin{cases} \int_{t'}^{t} dt & \text{if } t > t' \text{ on the contour } c \text{ and} \\ \int_c dt - \int_t^{t'} dt & \text{if } t < t' \text{ on the contour } c. \end{cases} \tag{4.10}$$

There are two ways we can deal with the Bosonic integrals. We shall present both as together they illuminate the interpretation of the result and allow for easier comparison with the work of Kamenev and Gefen[24]. Part of the Bosonic field can be dealt with exactly (equivalent to using a gauge transformation to remove it) and the remaining contribution from the field must be dealt with approximately.

The first approach we present uses the canonical ensemble representation to rewrite the integrals. If we define $\theta = \int_c dt\phi(t)$ then the Green's function, equation (4.9), can be written as

$$iG_n(t,t') = \mathrm{sgn}_c(t,t') \frac{\int D\phi e^{-\frac{i}{2E_c}\int_c \phi^2(t)dt} \mathcal{Z}_n(\theta) e^{\int_{c_{tt'}}[\phi(t)-i\xi_n]dt}}{\int D\phi e^{-\frac{i}{2E_c}\int_c \phi^2(t)dt} \mathcal{Z}(\theta)}, \tag{4.11}$$

where

$$\mathcal{Z}(\theta) = \prod_k \left[1 + e^{\theta - \beta\xi_k}\right] \text{ and} \tag{4.12}$$

$$\mathcal{Z}_n(\theta) = \prod_{k \neq n} \left[1 + e^{\theta - \beta\xi_k}\right]. \tag{4.13}$$

Note that in this case we have chosen to keep N_0 in the definition of the chemical potential. We can express $\mathcal{Z}_n$ and $\mathcal{Z}$ in terms of the canonical ensemble (instead of the grand canonical ensemble). We define

$$\mathcal{Z}(\theta) = \sum_{N=0}^{\infty} Z_N e^{(\beta\mu+\theta)N}, \qquad Z_N = \oint \frac{d\varphi}{2\pi} e^{-iN\varphi} \prod_k \left[1 + e^{-\beta\varepsilon_k + i\varphi}\right], \tag{4.14}$$

$$\mathcal{Z}_n(\theta) = \sum_{N=0}^{\infty} Z_N(\varepsilon_n) e^{(\beta\mu+\theta)N}, \qquad Z_N(\varepsilon_n) = \oint \frac{d\varphi}{2\pi} e^{-iN\varphi} \prod_{k \neq n} \left[1 + e^{-\beta\varepsilon_k + i\varphi}\right]. \tag{4.15}$$

Z_N is the canonical partition function for N particles. To show this it is simply necessary to expand the product and calculate the integral. $Z_N(\varepsilon_n)$ is the partition function without any N-particle states which contain the single particle level ε_n. More formally we can write

$$\frac{Z_N(\varepsilon_n)}{Z_N} = \frac{\mathrm{Tr}_N\, \hat{c}_n \hat{c}_n^\dagger e^{-\beta \hat{H}_0}}{\mathrm{Tr}_N\, e^{-\beta \hat{H}_0}} = 1 - F_N(\varepsilon_n) \tag{4.16}$$

with $F_N(\varepsilon_n)$ the distribution function for being in any N-particle state containing ε_n. As the charging energy is a constant in the canonical distribution it does not contribute to this expression.

Substituting the expression for $Z_N(\varepsilon_n)$ into the Green's function we can find equations (4.17) and (4.18). For a system with a large number of particles the canonical distribution function can be approximately replaced with the Fermi function, $F_N(\varepsilon - \omega_N) \approx f(\varepsilon - \omega_N)$. We also use the fact that Z_N is a smooth function, on a scale δ/T, and can be cancelled from the numerator and denominator.

The resulting expressions for the Green's functions are:

$$iG_n^<(\varepsilon) = -\frac{2\pi}{Z}\sum_{N=0}^{\infty} e^{-\beta E_N} f(\varepsilon - \omega_{N-1})\delta(\varepsilon_n - \varepsilon + \omega_{N-1}) \tag{4.17}$$

and

$$iG_n^>(\varepsilon) = \frac{2\pi}{Z}\sum_{N=0}^{\infty} e^{-\beta E_N}[1 - f(\varepsilon - \omega_N)]\delta(\varepsilon_n - \varepsilon + \omega_N). \tag{4.18}$$

The energy levels E_N are defined as $E_N = E_c(N - N_0)^2/2 - \mu N$. The difference in energy between consecutive energy levels is given by $\omega_N = E_{N+1} - E_N = E_c(N + \frac{1}{2}) - (\mu + N_0 E_c)$. The sum over N is explicitly shown to be a sum over N-particle states.

In order to compare where we differ from the calculation of Kamenev and Gefen it is more convenient to deal with a saddle point approximation in an analogous way to their method. It is not possible to use the exact same method as we cannot Fourier transform on the contour c to easily extract the zero frequency contribution of the bosonic field. To deal with the Bosonic integrals we define $\theta = \int_c dt\phi$ and demand that this quantity is real (we are free to do this by picking $\phi(t)$ to be real on the real time contour and imaginary on the imaginary time contour). Hence we can write

$$\int \frac{D\phi}{\mathcal{N}} e^{-\frac{i}{2E_c}\int_c dt\phi^2} F\Big(\phi(t), \int_c \phi(t)dt\Big) = \int \frac{D\phi}{\mathcal{N}} \int d\theta\delta\Big(\theta - \int_c \phi(t)dt\Big) \\ e^{-\frac{i}{2E_c}\int_c dt\{[\phi(t)+\theta/L]^2+[\theta/L]^2\}} F(\phi(t), \theta), \tag{4.19}$$

with $L = -\int_c dt = i\beta$. Now we can make the substitution $\tilde{\phi(t)} = \phi(t) + \theta/L$. We then find

$$\int_{-\infty}^{\infty} d\theta e^{-\frac{\theta^2}{2E_c\beta}} \int D\tilde{\phi}\delta\Big(\int_c \tilde{\phi}(t)dt\Big) e^{-\frac{i}{2E_c}\int_c dt\tilde{\phi}^2} F(\tilde{\phi}(t) + \theta/L, \theta) \tag{4.20}$$

We have also noted that the condition $\theta - \int_c dt\phi(t) = 0$ is identical to making $\int_c dt\tilde{\phi}(t) = 0$.

Applying this to the Green's function we can integrate over $\tilde{\phi}$ exactly. We have

$$iG_n(t,t') = \left\langle e^{\int_{c_{tt'}} \tilde{\phi}(t)dt} \right\rangle\Big|_{\int_c dt\tilde{\phi}(t)=0} \left\langle ig(t,t';\xi_n - \theta/\beta) \right\rangle_\theta . \tag{4.21}$$

The free particle Green's function may be derived in the functional integral representation:

$$ig_n(t,t') = \frac{\mathrm{sgn}_c(t,t')e^{-i\xi_n \int_{c_{tt'}} dt}}{1+e^{-\beta\xi_n}}. \tag{4.22}$$

Now the $\tilde{\phi}$ averaging is given by

$$\begin{aligned}\left\langle e^{\int_{c_{tt'}} \tilde{\phi}(t)dt} \right\rangle\Big|_{\int_c dt\tilde{\phi}(t)=0} &= \int \frac{D\tilde{\phi}}{\mathcal{N}} e^{-\frac{i}{2E_c}\int_c dt\tilde{\phi}^2(t)} e^{\int_{c_{tt'}} \tilde{\phi}(t)dt} \delta\Big(\int_c \tilde{\phi}(t)dt\Big) \\ &= e^{-\frac{E_c t^2}{2\beta} - \frac{iE_c t[\mathrm{sgn}_c(t)]}{2}}. \end{aligned} \tag{4.23}$$

This integral was performed in the same way as we removed the "zero-mode" part of the ϕ integral originally. We introduce a new variable to be integrated over, like θ, allowing us to perform the functional integral. By zero-mode we refer, in analogy to a zero Matsubara frequency mode, to $\int_c dt\phi(t)$. We are now left with an expression for the Green's function only in terms of the zero-mode.

The removal of all parts of the Bosonic field except the zero-mode term can also be achieved by a gauge transformation. This is what was done by Kamenev and Gefen originally[24]. The zero-mode term cannot be removed in this way however, for an explanation of why, see appendix F. However it is inconvenient for us to use this method as we can not Fourier transform on our time contour to trivially isolate this contribution.

The integral over θ will be performed in a saddle point approximation. After integrating over $\tilde{\phi}$ we are left with

$$iG_n(t) = e^{-\frac{E_c t^2}{2\beta} - \frac{iE_c t}{2} + iE_c t\theta_c(t)} \frac{\int d\theta e^{-\frac{\theta^2}{2E_c\beta} - N_0\theta + \ln(\mathcal{Z}(\theta))} ig(t;\xi_n - \theta/\beta)}{\int d\theta e^{-\frac{\theta^2}{2E_c\beta} - N_0\theta + \ln(\mathcal{Z}(\theta))}} \tag{4.24}$$

where, as before,

$$\mathcal{Z}(\theta) = \prod_k \left[1 + e^{\theta - \beta\xi_k}\right]. \tag{4.25}$$

The saddle point equation for the integral is therefore

$$0 = \frac{\theta_0}{\beta E_c} - N_0 + \sum_k f(\xi_k - \theta_0/\beta). \tag{4.26}$$

Now, as $\sum_k \sim T/\delta$ and N_0 are very large we can neglect the first term. δ is the mean level spacing and is assumed to be smaller than all other energy scales. Then making the substitution $\theta_0 = -\beta(\mu - \mu_0) + 2\pi m i$ we arrive at

$$N_0 = \sum_k f(\xi_k - \mu_0). \tag{4.27}$$

Note that there are an infinite number of saddle points and that μ_0 is acting like a chemical potential for a collection of N_0 electrons.

We can now find expressions for the various types of Green's functions. We now define $Z = \sum_N e^{-\beta E_N}$, where the chemical potential in $E_N = E_c N^2/2 - \tilde{\mu} N$ is now $\tilde{\mu} = \mu - \mu_0 + E_c N_0$. After Fourier transforming:

$$\begin{aligned} iG_n^<(\varepsilon) &= -\frac{2\pi}{Z} f(\xi_n + \mu - \mu_0) e^{-\frac{\beta}{2E_c}(\xi_n - \varepsilon - E_c/2 + \mu - \mu_0)^2 + \frac{\beta\tilde{\mu}^2}{2E_c}} \\ &\quad \sum_{N=0}^{\infty} \delta(E_c(N - N_0) + \xi_n - \varepsilon - E_c/2) \qquad (4.28) \\ iG_n^>(\varepsilon) &= \frac{2\pi}{Z}[1 - f(\xi_n + \mu - \mu_0)] e^{-\frac{\beta}{2E_c}(\xi_n - \varepsilon + E_c/2 + \mu - \mu_0)^2 + \frac{\beta\tilde{\mu}^2}{2E_c}} \\ &\quad \sum_{N=0}^{\infty} \delta(E_c(N - N_0) + \xi_n - \varepsilon + E_c/2). \qquad (4.29) \end{aligned}$$

Hence we have all the components of the Keldysh Green's functions using the relations:

$$\begin{aligned} &iG_n^R(t) = \theta(t)[iG_n^>(t) - iG_n^<(t)] \qquad iG_n^A(t) = -\theta(-t)[iG_n^>(t) - iG_n^<(t)] \\ &iG_n^K(t) = iG_n^>(t) + iG_n^<(t). \end{aligned} \tag{4.30}$$

To compare with the alternative derivation above it is more convenient to express the

Green's functions as:

$$iG_n^<(\varepsilon) = -\frac{2\pi}{Z}\sum_{N=0}^{\infty} e^{-\beta E_N} f(\varepsilon - \omega_{N-1})\delta(\varepsilon_n - \varepsilon + \omega_{N-1}) \text{ and} \tag{4.31}$$

$$iG_n^>(\varepsilon) = \frac{2\pi}{Z}\sum_{N=0}^{\infty} e^{-\beta E_N}[1 - f(\varepsilon - \omega_N)]\delta(\varepsilon_n - \varepsilon + \omega_N) \tag{4.32}$$

exactly as before by the properties of the delta function and $\omega_N = E_{N+1} - E_N = E_c(N + \frac{1}{2}) - \tilde{\mu}$. From now on we redefine $\tilde{\mu} \to \mu$ for ease of representation.

4.2 Results for the Isolated Quantum Dot

4.2.1 The Average Number of Particles

To find the average particle number from the Green's function we just need to use

$$\begin{aligned}\langle \hat{N} \rangle &= \langle \hat{\bar{\psi}}(0)\hat{\psi}(0) \rangle \\ &= -i\sum_n G_n(-0).\end{aligned} \tag{4.33}$$

We can use the Green's function in equation (4.24), substitute it into this expression, and then use the saddle point solution described above. First we will use the fact that

$$\begin{aligned}\frac{d\mathcal{Z}(\theta)}{d\theta} &= \sum_n \frac{1}{1 + e^{\beta\xi_n - \theta}}\mathcal{Z}(\theta) \text{ and therefore} \\ \langle N \rangle &= \frac{\int d\theta e^{-\frac{\theta^2}{2E_c\beta} - N_0\theta}\frac{d\mathcal{Z}(\theta)}{d\theta}}{\int d\theta e^{-\frac{\theta^2}{2E_c\beta} - N_0\theta}\mathcal{Z}(\theta)}.\end{aligned} \tag{4.34}$$

It is necessary to use this form to ensure our previous saddle point solution is valid. If we consider the saddle point to $\int d\theta e^{-S(\theta)} f(\theta)$, it is necessary for $f(\theta)$ to be "well-behaved", but this is not the case for our general Green's function. For a "well-behaved" function we require that it does not increase exponentially. After integrating by parts and using the saddle point approximation this gives

$$\langle N \rangle = \frac{\tilde{\mu}}{E_c} + \frac{2\pi i}{E_c\beta}\frac{\sum_m m e^{-\frac{(2\pi m)^2}{2E_c\beta} + \frac{i2\pi m\tilde{\mu}}{E_c}}}{\sum_m e^{-\frac{(2\pi m)^2}{2E_c\beta} + \frac{i2\pi m\tilde{\mu}}{E_c}}}. \tag{4.35}$$

From this we can calculate:

$$\langle N\rangle \approx \frac{\mu}{E_c} \quad \text{in the limit as } E_c \ll T \text{ and} \tag{4.36}$$

$$\langle N\rangle \approx \frac{\mu}{E_c} - 4\pi\frac{T}{E_c}e^{-\frac{T}{E_c}2\pi^2}\sin(2\pi\mu/E_c) \quad \text{in the limit } T \lesssim E_c. \tag{4.37}$$

This second result is the onset of the Coulomb staircase, i.e. at specific values of the chemical potential (changed by a gate voltage applied to the system) the number of particles "jumps" by one. Note however that more terms in the sum over m may be needed, depending on the ratio T/E_c. Figure 1.3 uses $T/E_c = 0.02$ and includes terms up to $m = 100$ terms. There is a trivial way of seeing this effect by looking for a solution to $H_c(N) = H_c(N+1)$, i.e. that it costs nothing to add an electron to the quantum dot. This occurs at values of the chemical potential given by $\mu = E_c(N + \frac{1}{2})$.

4.2.2 The Tunnelling Density of States

The density of states is given by the usual formula

$$\nu(\varepsilon) = \frac{i}{2\pi}\sum_n [G_n^R(\varepsilon) - G_n^A(\varepsilon)]. \tag{4.38}$$

If $\nu_0 \approx$constant it is then simple to see

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{1}{Z}\sum_N e^{-\beta E_N}[1 - f(\varepsilon-\omega_N) + f(\varepsilon-\omega_N)e^{-\beta\omega_N}] \tag{4.39}$$

$$E_N = \frac{E_c}{2}\left[N - \frac{\mu}{E_c}\right]^2 \tag{4.40}$$

$$\omega_N = E_{N+1} - E_N = E_c(N+1/2) - \mu. \tag{4.41}$$

$f(\varepsilon - \omega_N)$ are Fermi distribution functions.

We wish to maximize the coefficients ($e^{-\beta\omega_N}$ and $e^{-\beta E_N}$). It can be seen that the terms we need to keep are those of order E_{M_0} and E_{M_0+1}. (M_0 is the maximal term and we can ignore $M_0 - 1$ as it is exponentially suppressed.) Note, as shall be seen below, M_0 shall be the closest integer to μ/E_c.

The Density of States in the Valley

In the centre of the Coulomb valley, $\mu = M_0 E_c$, we find, for $\beta E_c \gg 1$,

$$\frac{\nu(\varepsilon)}{\nu_0} = 1 - f(\varepsilon - \omega_{M_0}) + f(\varepsilon - \omega_{M_0 - 1}) \text{ where} \tag{4.42}$$

$$\omega_N = \big(N - M_0 + \frac{1}{2}\big) E_c. \tag{4.43}$$

This describes the standard gap in the tunnelling density of states pictured in diagram 4.3. In the Coulomb valley there are no states to be tunnelled into and hence the conductance is suppressed around these points.

The Density of States at the Degeneracy Point

Near the degeneracy point, $\mu = E_c(M_0 + 1/2) + \delta\mu$, we find, for $\beta E_c \gg 1$, that

$$\frac{\nu(\varepsilon)}{\nu_0} = \frac{1 - f(\varepsilon + \delta\mu) + f(\varepsilon + E_c + \delta\mu)}{1 + e^{\beta\delta\mu}} \tag{4.44}$$

$$+ \frac{e^{\beta\delta\mu}[1 - f(\varepsilon - E_c + \delta\mu) + f(\varepsilon + \delta\mu)]}{1 + e^{\beta\delta\mu}}. \tag{4.45}$$

This gives the cases, for $|\delta|\mu \gg T$,

$$\frac{\nu(\varepsilon)}{\nu_0} \approx 1 + \frac{1}{2}[f(\varepsilon + E_c) - f(\varepsilon - E_c)] \text{ if } \delta\mu = 0, \tag{4.46}$$

$$\frac{\nu(\varepsilon)}{\nu_0} \approx 1 - f(\varepsilon + \delta\mu - E_c) + f(\varepsilon + \delta\mu) \text{ if } \delta\mu > 0, \tag{4.47}$$

$$\frac{\nu(\varepsilon)}{\nu_0} \approx 1 + f(\varepsilon + \delta\mu + E_c) - f(\varepsilon + \delta\mu) \text{ if } \delta\mu < 0. \tag{4.48}$$

At the degeneracy point ($\delta\mu = 0$) the gap in the density of states is reduced to a "half-gap", see figure 4.1. As we shall see this half-gap is required to regain the correct form for the differential conductance at the degeneracy points[19, 20, 22, 21].

4.3 Current Through a Dot

We wish to consider a system comprised of two quasi-one dimensional leads attached to a quantum dot by point contacts. The quantum dot is described by the Hamiltonian

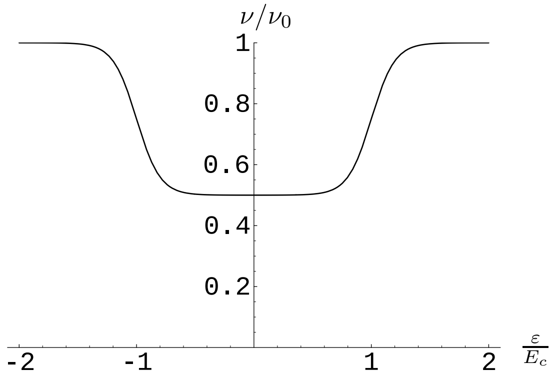


Figure 4.1: The density of states at $T \neq 0$ at the degeneracy point.

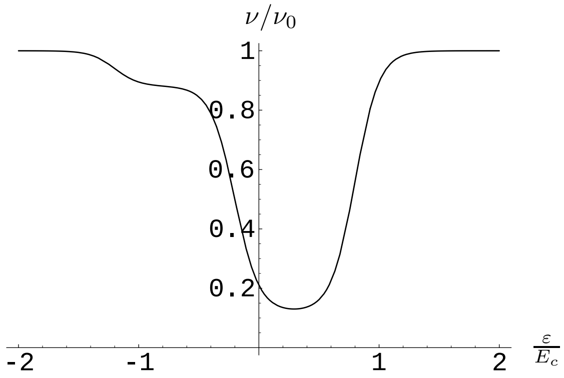


Figure 4.2: The density of states at $T \neq 0$ near the degeneracy point. The intermediate case between figures 4.1 and 4.3

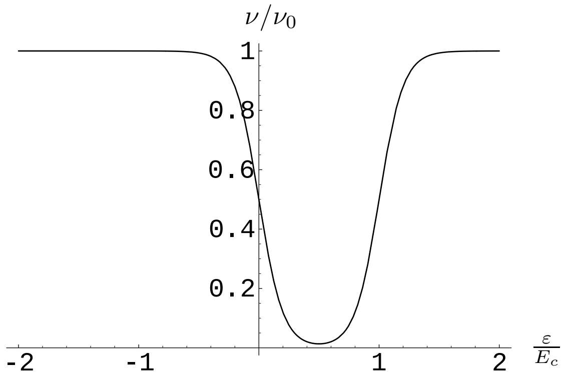


Figure 4.3: The density of states at $T \neq 0$ away from the degeneracy point.

$\hat{H} = \hat{H}_0 + \hat{H}_i$ as before, see equation (4.1). However we now also include the leads in $\hat{H}_0$ and introduce a coupling term, $\hat{H}_t+$, between the dot and the leads. The current through the dot is given by

$$I = \dot{Q} = ei[\hat{H}, \hat{N}] = ei[\hat{H}_t, \hat{N}] \tag{4.49}$$

$$\hat{H} = \hat{H}_0 + \hat{H}_t + \hat{H}_i \tag{4.50}$$

$$\hat{H}_t = \sum_{\alpha,n,k} [t_{\alpha,n,k} \hat{d}^{\dagger}_{\alpha,n} \hat{a}_k + t^*_{\alpha,n,k} \hat{a}^{\dagger}_k \hat{d}_{\alpha,n}]. \tag{4.51}$$

Where α labels the leads (left and right), $\hat{a}^{\dagger}$ is the creation operator for electrons in the dot and $\hat{d}^{\dagger}_{\alpha}$ is the creation operator for electrons in lead α. Calculating the commutator in the definition of the current, I, tells us that

$$I = ie \sum_{\alpha,n,k} f_{\alpha} \langle t_{\alpha,n,k} d^{\dagger}_{\alpha,n} a_k - t^*_{\alpha,n,k} a^{\dagger}_k d_{\alpha,n} \rangle \text{ with} \tag{4.52}$$

$$f_{\alpha} = \begin{cases} 1 & \text{left lead} \\ -1 & \text{right lead} \end{cases}. \tag{4.53}$$

f_{α} is introduced to reflect the fact that tunnelling between the left lead and the dot is in the opposite direction to tunnelling between the right lead and the dot.

The current can be written using the functional integral representation and introducing a source function, J. If we have

$$I = e \sum_{\alpha,n,k} \left. \frac{\partial \ln Z[J]}{\partial [J_{\alpha,n,k}(t)]} \right|_{J=0} \text{ with} \tag{4.54}$$

$$Z[J] = \int D\psi D\bar{\psi} e^{iS_0 + iS_t + i\sum_{\alpha,n,k} f_{\alpha} \int_c dt [t_{\alpha,n,k} \psi_k(t) \bar{\psi}_{\alpha,n}(t) - t^*_{\alpha,n,k} \psi_{\alpha,n}(t) \bar{\psi}_k(t)] J_{\alpha,n,k}(t)}, \tag{4.55}$$

then performing the derivative gives us

$$I = \frac{ie}{Z[0]} \sum_{\alpha,n,k} f_{\alpha} \int D\psi D\bar{\psi} \big[t_{\alpha,n,k} \psi_k(t) \bar{\psi}_{\alpha,n}(t) - t^*_{\alpha,n,k} \psi_{\alpha,n}(t) \bar{\psi}_k(t) \big] e^{iS_0 + iS_t}. \tag{4.56}$$

This is the functional integral representation of equation (4.52). The actions in equation

(4.54) are

$$iS_0 = i\int_c dt[\sum_k \bar{\psi}_k(t)i\partial_t\psi_k(t) - H] + i\sum_{n,\alpha}\int_c dt\bar{\psi}_{\alpha,n}(t)[i\partial_t - \xi_{n,\alpha}]\psi_{\alpha,n}(t) \text{ and} \tag{4.57}$$

$$iS_t = i\sum_{\alpha,n,k}\int_c dt[t_{\alpha,n,k}\bar{\psi}_{\alpha,n}(t)\psi_k(t) + t^*_{\alpha,n,k}\bar{\psi}_k(t)\psi\alpha, n(t)]. \tag{4.58}$$

We imply that $D\psi D\bar{\psi}$ are the integrals over the fields for both leads ($\{\psi_{\alpha,n}\}$) and the dot ($\{\psi_k\}$). H is the Hamiltonian for the quantum dot given in equation (4.1) and $\xi_{n,\alpha}$ are the dispersion relations for the lead electrons.

The next step is to integrate out the leads from this expression leaving the Green's function for the dot coupled to two mass operators courtesy of the leads. We define the mass operators as

$$i\Sigma_{\alpha,k,k'}(t,t') = \sum_n t_{\alpha,n,k}t^*_{\alpha,n,k'}iG_{\alpha,n}(t,t') = \sum_n t_{\alpha,n,k}t^*_{\alpha,n,k'}\langle \mathrm{T}\,\psi_{\alpha,n}(t)\bar{\psi}_{\alpha,n}(t')\rangle, \tag{4.59}$$

which leads to

$$I = e\sum_{\alpha,kk'} f_\alpha \int_c dt'[iG_{kk'}(t,t')i\Sigma_{\alpha,k'k}(t',t) - i\Sigma_{\alpha,kk'}(t,t')iG_{k'k}(t',t)] \tag{4.60}$$

for the current. The Green's function for the quantum dot coupled to the leads is then

$$iG_{kk'}(t,t') = \frac{1}{Z}\int D\psi D\bar{\psi}\psi_k(t)\bar{\psi}_{k'}(t')e^{iS_0 - i\sum_{\alpha,kk'}\int_c dtdt'\bar{\psi}_k(t)\Sigma_{\alpha,kk'}(t,t')\psi_{k'}(t')}. \tag{4.61}$$

We can now convert our contour to the usual Keldysh contour by extending it to plus and minus infinity and neglecting the "tail" ($t_0 \to t_0 - i\beta$). Splitting the Green's functions into contributions from $t < t'$ and vice versa and rearranging gives

$$I = e\sum_{\alpha,kk'} f_\alpha \int_{-\infty}^{\infty} dt'[iG^>_{kk'}(t,t')i\Sigma^<_{\alpha,k'k}(t',t) - i\Sigma^>_{\alpha,kk'}(t,t')iG^<_{k'k}(t',t)]. \tag{4.62}$$

Which, due to all the Green's functions being diagonal in frequency space as shall be seen, can be arranged to give

$$I = \frac{e}{2}\sum_{\alpha,kk'} f_\alpha \int_{-\infty}^{\infty} dt'[iG^k_{kk'}(t-t')\Delta\Sigma_{\alpha,k'k}(t'-t) - i\Sigma^k_{\alpha,kk'}(t-t')\Delta G_{k'k}(t'-t)]. \tag{4.63}$$

Where we have used the notation $\Delta G \equiv iG^R - iG^A$.

Dyson's equation can be found from equation (4.61). With g as the uncoupled dot and defining $\Sigma \equiv \Sigma_L + \Sigma_R$ we find $(G)^{-1} = (g)^{-1} - \Sigma$, which can be written explicitly as

$$\begin{aligned} iG^{R/A}_{nm}(\omega) &= ig^{R/A}_{nm}(\omega) - ig^{R/A}_{no}(\omega)i\Sigma^{R/A}_{ol}(\omega)iG^{R/A}_{lm}(\omega) && (4.64)\\ iG^K_{nm}(\omega) &= ig^K_{nm}(\omega) - [ig^R_{no}(\omega)i\Sigma^R_{ol}(\omega)iG^K_{lm}(\omega) \\ &\quad +ig^R_{no}(\omega)i\Sigma^K_{ol}(\omega)iG^A_{lm}(\omega) + ig^K_{no}(\omega)i\Sigma^A_{ol}(\omega)iG^A_{lm}(\omega)]. && (4.65) \end{aligned}$$

The indices o and l are summed over. It is important to note that the quantum dot described by g is the full interacting Green's function calculated in section 4.1. Now as Σ_{nm} is diagonal we can trivially solve Dyson's equation. This is true as we assume that the probability of tunnelling to different energy levels is uncorrelated[8].

Using these definitions along with

$$\begin{aligned} \sum_k ig^{R/A}_k(\omega) &= \pm\pi\nu(\omega) && (4.66)\\ i\Sigma^{R/A}_{nm} &= \pm\frac{1}{2}(\underbrace{\Gamma_L + \Gamma_R}_{\equiv\Gamma})\delta_{nm} && (4.67)\\ \Gamma_\alpha &= 2\pi\nu_\alpha|t|^2, && (4.68) \end{aligned}$$

we find for the current, to lowest order in $\nu(\omega)\Gamma$,

$$I = \frac{e}{2}\sum_\alpha f_\alpha \int \frac{d\omega}{2\pi}(h - h_\alpha)2\pi\nu(\omega)\Gamma_\alpha \tag{4.69}$$

The distribution function for the coupled dot, $h(\omega)$, is then found by balancing currents through the left and right contact, as in a steady state there should be no build up of charge in the dot. This condition can be written as $I_L = I_R$ where $I = I_L + I_R$ and leads to

$$h = \frac{h_L\Gamma_L + h_R\Gamma_R}{\Gamma_L + \Gamma_R}. \tag{4.70}$$

As the leads are in well defined equilibria $h_\alpha(\omega) = \tanh[(\omega - \mu_\alpha)/2T]$ with the chemical potential for the leads including a bias voltage: $\mu_\alpha = \mu - eV_\alpha$. We can also derive this consideration from the Kinetic equation for a region coupled to two reservoirs (the leads), see section 4.3.1.

Thus we find

$$I = \frac{e}{2}\int \frac{d\omega}{2\pi}[h_R(\omega) - h_L(\omega)]\frac{\Gamma_L\Gamma_R}{\Gamma_L + \Gamma_R}2\pi\nu(\omega). \tag{4.71}$$

For a central region without interaction this is just the two channel Landauer formula. If we rewrite equation(4.63) using $\Delta G = -iG^R iG^A \Delta\Sigma = -iG^R iG^A \Gamma$ we find

$$I = \frac{e}{2}\int \frac{d\omega}{2\pi}[h_R(\omega) - h_L(\omega)]\,\mathrm{tr}[T(\omega)T^\dagger(\omega)] \text{ where} \tag{4.72}$$

$$T_{\alpha\alpha'}(\omega) = 2\pi\sqrt{\nu_L}\sqrt{\nu_R}t^*_\alpha t_{\alpha'}\sum_{nm} G^R_{nm}(\omega). \tag{4.73}$$

See, for example, Meir and Wingreen[29, 30] and Imry and Landauer[44].

Taking the expression for the density of states in the regime where $E_c\beta \gg 1$ we find, to linear order in the bias $V = V_L - V_R$,

$$\begin{aligned} I = \frac{e^2V\nu_0}{2}\frac{\Gamma_L\Gamma_R}{\Gamma_L + \Gamma_R}\int dx\frac{\mathrm{sech}^2(x)}{(1+e^{\beta\delta\mu})}\Bigg[1 + \frac{1}{2}\tanh\left(x + \frac{\delta\mu - E_c}{2T}\right) - \frac{1}{2}\tanh\left(x + \frac{\delta\mu}{2T}\right) \\ + e^{\beta\delta\mu}\left\{1 + \frac{1}{2}\tanh\left(x + \frac{\delta\mu + E_c}{2T}\right) - \frac{1}{2}\tanh\left(x + \frac{\delta\mu}{2T}\right)\right\}\Bigg] \end{aligned} \tag{4.74}$$

This will then lead to

$$\begin{aligned} G = \frac{dI}{dV} = \frac{e^2\nu_0}{2(1+e^{\beta\delta\mu})}\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\Bigg[\left\{2 - \coth(\delta\mu\beta/2) + \frac{\delta\mu\beta}{2}\mathrm{csch}^2(\delta\mu\beta/2)\right\} \\ + e^{\beta\delta\mu}\left\{2 + \coth(\delta\mu\beta/2) - \frac{\delta\mu\beta}{2}\mathrm{csch}^2(\delta\mu\beta/2)\right\}\Bigg]. \end{aligned} \tag{4.75}$$

Leading to, for small $\delta\mu$,

$$G = e^2\nu_0 \frac{\Gamma_L\Gamma_R}{\Gamma_L + \Gamma_R} \frac{\delta\mu\beta/2}{\sinh(\delta\mu\beta)/2}. \tag{4.76}$$

This is the classic result of a peak in conductance at the degeneracy point which is exponentially suppressed in the valley. It is worth noting that we require a correct description of the density of states at (and near) the degeneracy point to correctly describe this behaviour.

4.3.1 The Kinetic Equation

We can derive the expression for the steady state distribution function for a central region coupled to two reservoirs. We use the Keldysh component of Dyson's equation which is equivalent to the kinetic equation[45]. We assume the uncoupled dot is completely isolated, therefore it has a fixed number of particles and is neither in any sort of equilibrium nor in the grand canonical ensemble. The leads are assumed to be in well defined equilibria. Coupling between the leads and the dot is switched on at some time in the past and the system will then reach a steady state. This switching process will be described by a Kinetic equation.

Let us solve the kinetic equation for a trivial set up. We will assume a "dot" with a single state coupled to two leads as before. To find the kinetic equation we start from the Keldysh component of Dyson's equation:

$$iG^K = ig^K - ig^R i\Sigma^R iG^K - ig^R i\Sigma^K iG^A - ig^K i\Sigma^A iG^A. \tag{4.77}$$

Matrix multiplication is implied in all necessary indices. If we also use the other two components of Dyson's equation we can rearrange this to give

$$(iG^R)^{-1} iG^K (iG^A)^{-1} = (ig^R)^{-1} ig^K (ig^A)^{-1} - i\Sigma^K. \tag{4.78}$$

We wish to solve this for $\mathcal{F}_{tt'}$, the distribution function for the dot. The coupling ($t_{\alpha,n,k}$) between the leads and the dot will be time dependent: at $(t - t') = -\infty$ the dot is assumed to be uncoupled from the leads and then the coupling is slowly turned on. At

$(t-t') = 0$ the leads and the dot are coupled. We then require the behaviour of $\mathcal{F}_{tt'}$ at $(t-t') = \infty$ when the system has reached a steady state. We can write the ansatz $iG^K = iG^R\,\mathcal{F} - \mathcal{F}\,iG^A$, with matrix multiplication over the time indices important in the order. Substituting in the operator forms $(iG^{R/A})^{-1} = (\pm\partial_t + i\xi)\delta_{tt'} + i\Sigma^{R/A}_{tt'}$ leads us to

$$\partial_{t+t'}\,\mathcal{F}_{t't} + i\Sigma^R_{tt_1}\,\mathcal{F}_{t_1t'} - \mathcal{F}_{tt_1}\,i\Sigma^A_{t_1t'} = i\Sigma^K - (ig^R)^{-1}ig^K(ig^A)^{-1}. \tag{4.79}$$

When this has reached a steady state, and the couplings between dot and leads are constant, it is simple to see

$$\mathcal{F}(\omega) = \frac{h_L(\omega)\Gamma_L + h_R(\omega)\Gamma_R}{\Gamma_L + \Gamma_R}. \tag{4.80}$$

ω is the Fourier transform of $t-t'$. In a steady state there will be no dependence on $t+t'$. Note that $(ig^{R/A})^{-1} = (\pm\partial_t + i\xi \mp i\delta)\delta_{tt'}$ and so $(ig^R)^{-1}ig^K(ig^A)^{-1} = \partial_{t+t'}h_{tt'} = 0$ in the steady state.

4.3.2 Nonlinear Conductance

We can also find the expression for nonlinear conductance. Starting from

$$I = \frac{e}{2}\int\frac{d\omega}{2\pi}[h_R(\omega) - h_L(\omega)]\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}2\pi\nu(\omega) \tag{4.81}$$

we find

$$\begin{aligned} I = {} & \frac{e\nu_0}{2}\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\frac{2T}{1+e^{\beta\delta\mu}}\int dx\left[\tanh\left(x - \frac{eV_R}{2T}\right) - \tanh\left(x - \frac{eV_L}{2T}\right)\right] \\ & \times\left[1 + \frac{1}{2}\tanh\left(x + \frac{\delta\mu - E_c}{2T}\right) - \frac{1}{2}\tanh\left(x + \frac{\delta\mu}{2T}\right)\right. \\ & \left. + e^{\beta\delta\mu}\left\{1 + \frac{1}{2}\tanh\left(x + \frac{\delta\mu + E_c}{2T}\right) - \frac{1}{2}\tanh\left(x + \frac{\delta\mu}{2T}\right)\right\}\right] \end{aligned} \tag{4.82}$$

near the degeneracy point. Performing the integral then leaves the result

$$\begin{aligned}I = {} & \frac{e\nu_0}{2}\frac{\Gamma_L\Gamma_R}{\Gamma_L+\Gamma_R}\bigg[e(V_L - V_R) + \tanh(\delta\mu\beta/2) \\ & \times\bigg\{(eV_R + \delta\mu)\coth[(eV_R+\delta\mu)\beta/2] \\ & -(eV_L+\delta\mu)\coth[(eV_L+\delta\mu)\beta/2)]\bigg\}\bigg]. \end{aligned} \tag{4.83}$$

This recovers the previous result in the linear conductance regime. This result does not depend exclusively on $V = V_L - V_R$, as the distribution functions alter differently near the regions V_R and V_L. Exactly at the degeneracy point the conductance is linear and recovers the previous result.

4.4 Summary

In this chapter we have derived the Green's function for an isolated quantum dot and compared this to previous derivations. We then applied this to the case of a dot weakly coupled to two leads and looked at the differential conductance through the system.

Chapter 5

BACKGROUND AND MOTIVATION FOR METAMAGNETIC CRITICALITY

In this chapter we present the concept of a quantum phase transition and discuss a particular example. We are interested in the metamagnetic phase transition seen in the bilayer ruthenate $Sr_3Ru_2O_7$. We will present the experimental evidence for such a quantum phase transition and introduce the background theory for this system. In the following chapter we shall address a specific model for this sample.

5.1 Quantum Critical Points

The idea of a zero temperature phase transition was first worked out by Hertz[46] in 1976. However it was not until the early nineties when the issue was taken up again and it has now become a very active area of study[47, 48, 49]. In a normal second order phase transition at finite temperatures a phase plot can look like figure 5.1. At the critical point of a phase transition there are two coexistent and competing phases. The idea of a quantum critical phase transition is to tune the end point of the line of phase transitions to zero temperature. This is achieved by varying some part of the system such as pressure or, in $Sr_3Ru_2O_7$, the angle of a magnetic field applied to the sample. Once the end point is lying on the zero temperature axis it is referred to as a quantum critical endpoint. The correlations associated with this transition will control the behaviour around the quantum critical point. A quantum phase transition is when the ground state of a system undergoes a phase transition, at zero temperature, as some parameter of the system is

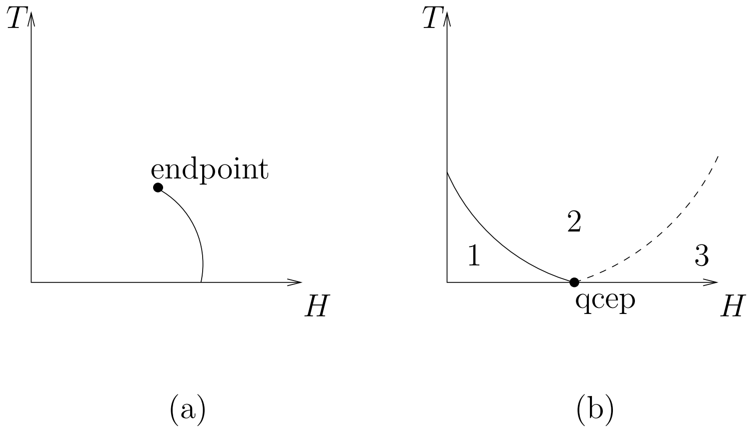


Figure 5.1: (a) A second order phase transition with a critical endpoint. Or, in the case of the metamagnetic transition, a first order phase transition. (b) Tuning a parameter, for example the direction of the field relative to the c-axis in $Sr_3Ru_2O_7$, forces the endpoint to $T = 0$ giving a quantum critical endpoint. H is applied magnetic field. Region 1 is the classically ordered state and region 3 is the quantum disordered state. In between these at finite temperature is region 2, dominated by the critical fluctuations of the quantum critical point. For the metamagnetic transition both 1 and 3 are Fermi metals. (qcep stands for the quantum critical endpoint.)

changed. As we are at zero temperature it is driven by quantum, rather than thermal, fluctuations. This parameter could be chemical composition, magnetic field, pressure or some other variable of the system.

We are interested in the case of the metamagnetic quantum critical point which is manifested in $\mathrm{Sr_3Ru_2O_7}$. The structure of $\mathrm{Sr_3Ru_2O_7}$ is sketched in figure 5.4. It is a bilayer ruthenate. (The single layer version is an unconventional superconductor.) It is the $\mathrm{RuO_2}$ layers that are active in the a-b plane, and we are interested in transport properties in these layers. The behaviour of the resistivity and magnetic susceptibility is what suggests a quantum critical point in this material.

A metamagnetic material is one which demonstrates a sharp rise in magnetization at a specific applied magnetic field, see figure 5.5. Before this jump in magnetization they act as paramagnets. This transition is a first order phase transition. At low enough temperatures, below about 5K, $Sr_3Ru_2O_7$ exhibits this kind of behaviour, see figure 5.2. If the applied B-field is in the ab plane of $Sr_3Ru_2O_7$ then this occurs at around 5 Tesla. Strictly speaking for a metamagnetic transition to occur the jump in magnetization must be sudden. This discontinuity shows up as a divergence in the magnetic susceptibility.

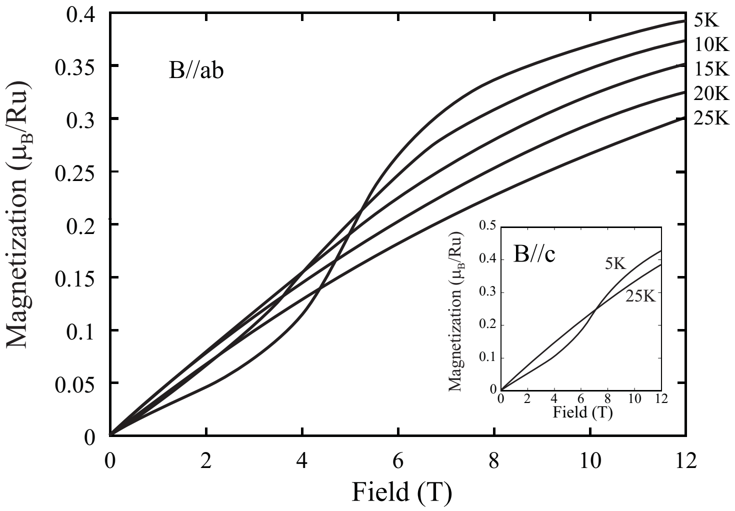


Figure 5.2: Data showing the onset of a metamagnetic phase transition as temperature is lowered, from[50]. Also shown is the reduction of the metamagnetic transition when the magnetic field is parallel to the c-axis.

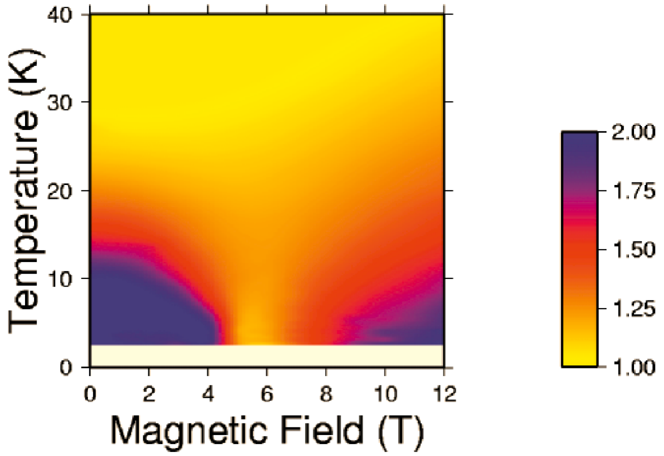


Figure 5.3: A coded plot showing T dependence of resistivity, from [50]. The shades label the power, α, in $\rho = \rho_0 + AT^{\alpha}$.

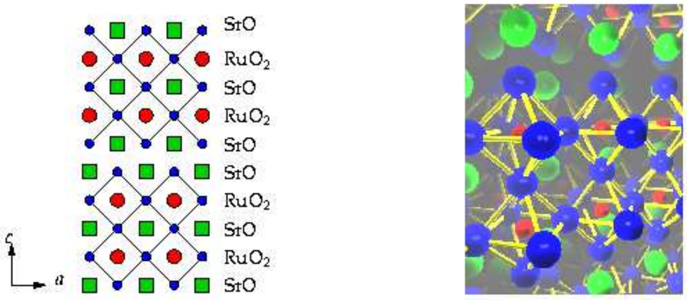
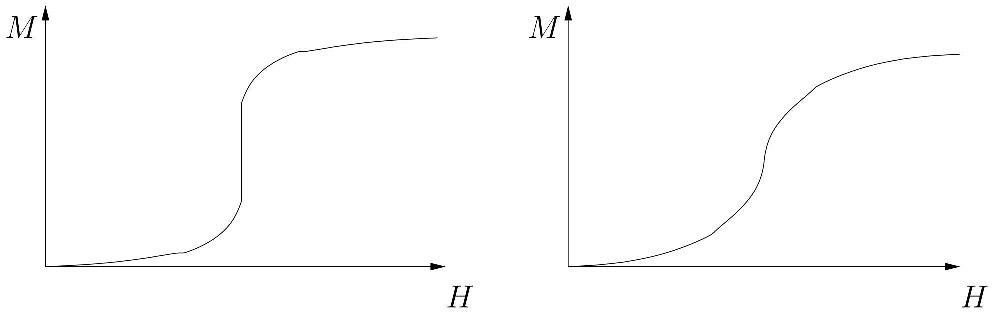


Figure 5.4: Structure of $Sr_3Ru_2O_7$. Courtesy of [51].



(a) (b)

Figure 5.5: These schematic diagrams show the metamagnetic phase transition. Diagram (a) demonstrates the sudden jump in magnetization, M, at a certain value of the applied magnetic field, B. The slope of the magentization, which is related to the susceptibility, diverges at the transition. Diagram (b) shows this transition just vanishing at the end point when the gradient just diverges but there is no sudden jump in the magnetization.

The line of phase transitions in figure 5.1(a) is the line of these transitions. The end point is the position in the phase diagram at which the sudden jump in the magnetization disappears as we rotate the orientation of the B-field with respect to the c-axis of the system. As the angle of B-field to the c-axis changes it is this end point which is tuned down to $T = 0$. The end point falls below the scope of experiment at an angle of $5\,°$. The residual line of finite temperature phase transitions is then indistinguishable from a quantum critical point.

The simplest data which demonstrates the reason for believing a quantum critical point exists in $Sr_3Ru_2O_7$ is a “phase plot”. Figure 5.3 shows the temperature dependence of the resistivity, $\rho = \rho_0 + AT^\alpha$, for different applied $B_{\|c}$ and over ranges of T. This is found from

$$\frac{\partial \ln(\rho - \rho_0)}{\partial(\ln T)} = \alpha. \tag{5.1}$$

So to find α, $\ln(\rho - \rho_0)$ is plotted against $\ln T$ and the gradient is calculated. As can be seen from the plot, this pictorially represents exactly what would be expected of a quantum critical point. (Compare figure 5.3 to figure 5.1(b).) The areas where $\alpha = 2$ are the standard Fermi liquid behaviour. The linear T dependence of the resistivity at higher temperatures is controlled by the quantum critical point[52]. When the critical field is aligned with the c-axis a novel resistivity occurs as $T \to 0$, $\alpha \to 3$. This low temperature state is driven by the divergent fluctuations of the quantum critical point.

A clear way of looking for the transition is to study the magnetic susceptibility, Ξ. From

$$\Xi = \left.\frac{\partial M}{\partial H}\right|_{T,p}, \tag{5.2}$$

with M the magnetization and H the applied field, a metamagnetic transition will experimentally appear as a maximum in the susceptibility. Theoretically the endpoint of the transition will give a divergent cusp in the susceptibility. These can be measured experimentally[52, 53, 54]. Figure 5.6 demonstrates the appearance of these cusps.

It is possible to experimentally follow the line of critical endpoints down toward zero with changing field direction and strength[53]. Figure 5.7 shows the temperature of the critical endpoint being tuned to zero temperature by changing the direction of the magnetic field with the a-b planes. The value of the critical field at which the transition occurs can also be followed as we approach the quantum critical point: see figure 5.8.

The information in these plots can be collected into a 3-d phase diagram, figure 5.9. The region which is shaded shows all peaks in the magnetic susceptibility.

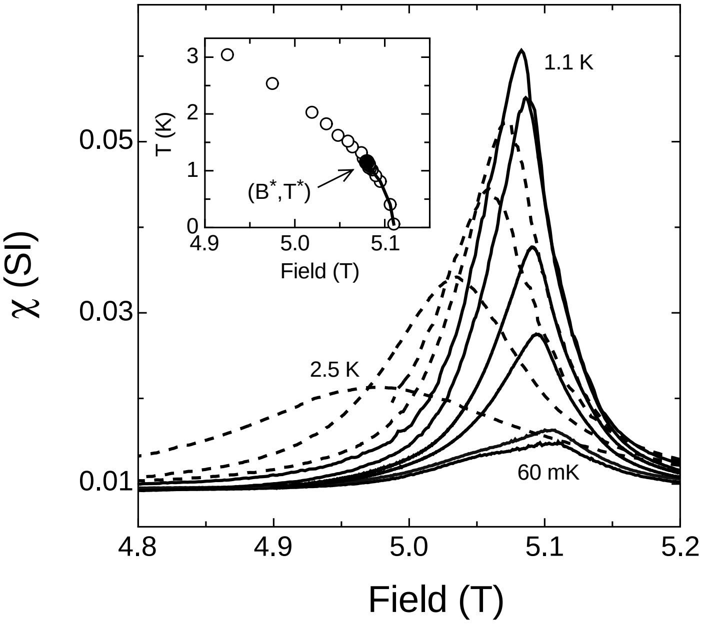


Figure 5.6: Magnetic susceptibility in $Sr_3Ru_2O_7$[54].

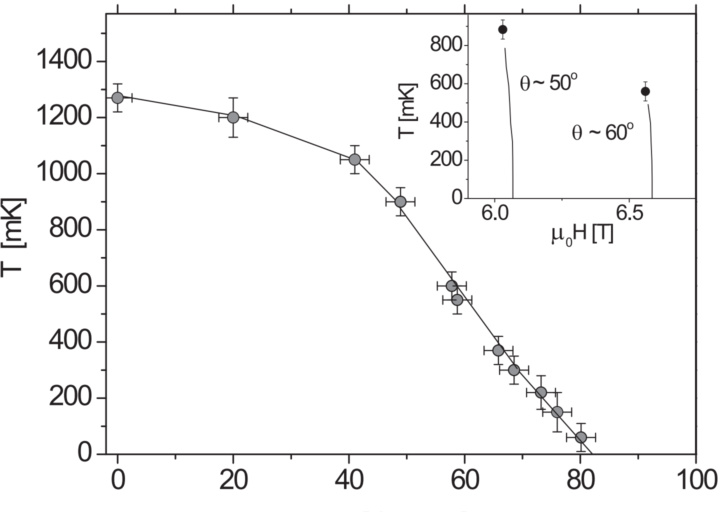


Figure 5.7: The critical temperature in $Sr_3Ru_2O_7$ as a function of the angle between the B-field and the a-b planes[53].

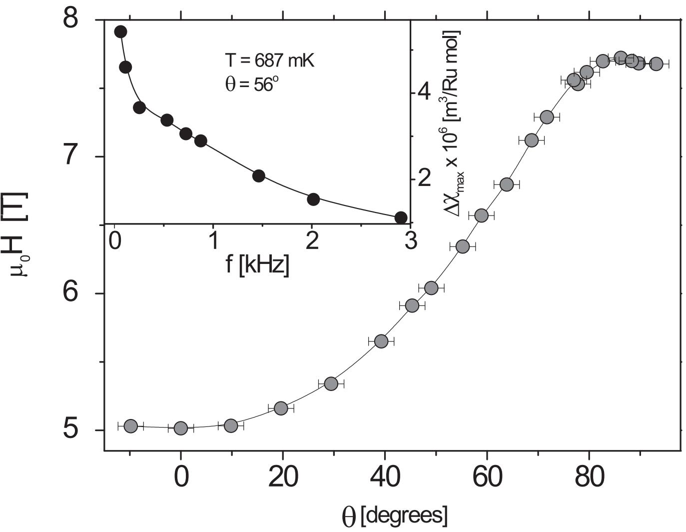


Figure 5.8: The critical field in $Sr_3Ru_2O_7$ as a function of the angle between the B-field and the a-b planes[53].

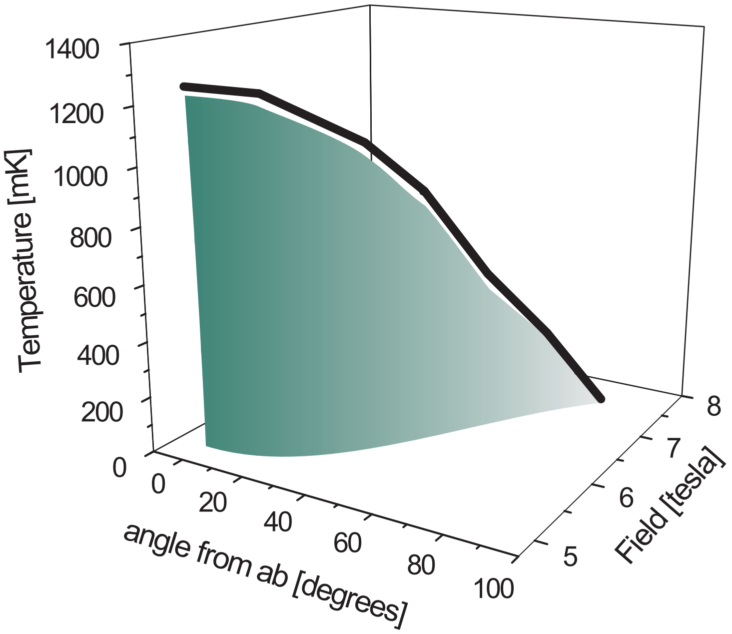


Figure 5.9: The line of endpoints in $Sr_3Ru_2O_7$[53].

5.2 Quantum Criticality from the Hubbard Interaction

Firstly we will introduce the basic idea that Hertz proposed in his 1976 paper on zero temperature phase transitions. Starting from the Hubbard interaction, Hertz[46] derived a Ginzburg-Landau functional concentrating on the spin-density fluctuations. With the purpose of viewing the zero, or low temperature, quantum critical phenomena. The Hubbard interaction is given by

$$\hat{H}_{int} = V \sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}, \tag{5.3}$$

with the spin dependent number operator $\hat{n}_{i\sigma} = \hat{\psi}^\dagger_{i\sigma}\hat{\psi}_{i\sigma}$. This can be expanded into the charge, $\hat{n}^c_i$, and spin, $\hat{n}^s_i$, fluctuations separately.

$$\hat{H}_{int} = \frac{V}{4}\sum_i \underbrace{(\hat{n}_{i\uparrow} + \hat{n}_{i\downarrow})^2}_{=\hat{n}^c_i} - \frac{V}{4}\sum_i \underbrace{(\hat{n}_{i\uparrow} - \hat{n}_{i\downarrow})^2}_{=\hat{n}^s_i}. \tag{5.4}$$

We expect only the spin fluctuations to be important, and henceforth the charge fluctuations of the interactions shall be neglected. This can be strictly checked by including these terms and verifying that they are small in the expansion.

We start from a functional integral of the continuous limit of the interaction given by the spin contribution of equation (5.4). This gives the partition function as

$$Z = \int D\psi D\bar{\psi} e^{-\int_0^\beta d\tau \int d\mathbf{r} \sum_\sigma \bar{\psi}_\sigma(x)[\partial_\tau + \hat{\xi}(\mathbf{r})]\psi_\sigma(x)} e^{\frac{V}{4}\int_0^\beta d\tau \int d\mathbf{r}[n^s(x)]^2}. \tag{5.5}$$

Performing a Hubbard-Stratonovich transformation on the interacting part, such that

$$\frac{1}{\mathcal{N}}\int D\phi e^{-\frac{1}{V}\int d^3x\phi^2(x)} e^{-\int d\tau d\mathbf{r}\phi(x)n^s(x)} = e^{\frac{V}{4}\int d\tau d\mathbf{r}[n^s(x)]^2}, \tag{5.6}$$

will give

$$Z = \int D\phi e^{-\frac{1}{V}\int d^3x\phi^2} \int D\psi d\bar{\psi} e^{-\sum_\sigma \int d^3x \bar{\psi}_\sigma(x)(\partial_\tau + \hat{\xi}(\mathbf{r}) + \sigma\phi(x))\psi_\sigma(x)}. \tag{5.7}$$

Now performing the Gaussian fermionic integrals and using the result $\ln[\det \mathbf{M}] = \mathrm{Tr}\ln[\mathbf{M}]$

$$Z = \int D\phi e^{-\frac{1}{V}\int d^3x\phi^2(x)} e^{\mathrm{Tr}\ln[\partial_\tau+\xi(\mathbf{r})+\phi(x)]} e^{\mathrm{Tr}\ln[\partial_\tau+\xi(\mathbf{r})-\phi(x)]}. \tag{5.8}$$

We need to evaluate the logarithmic term. This shall be done perturbatively.

If we write $G(x) = (\partial_\tau + \xi(\mathbf{r})^{-1}$ then we can rewrite Z as, ignoring any constant coefficients,

$$Z = \int D\phi e^{-\frac{1}{V}\int d^3x\phi^2(x)} e^{\mathrm{Tr}\ln[1+G(x)\phi(x)]} e^{\mathrm{Tr}\ln[1-G(x)\phi(x)]}. \tag{5.9}$$

We can now expand the two logarithmic terms in powers of "$G\phi$". This will lead to a Ginzburg-Landau type functional with ϕ playing the role of the order parameter. What this order parameter corresponds to shall be expounded shortly. Firstly, if we assume that we can truncate the expansion then, symbolically,

$$Z \approx \int D\phi e^{-\frac{1}{V}\int d^3x\phi^2(x)+\mathrm{Tr}[-G^2\phi^2-G^4\phi^4]}. \tag{5.10}$$

The second order term in this expansion contains the correlator χ_0. In the Matsubara representation this correlator is

$$\chi_0(\mathbf{q},\omega_n) = -T\sum_{k,\epsilon_m} G(\mathbf{k},\epsilon_m)G(\mathbf{k}+\mathbf{q},\omega_n+\epsilon_m). \tag{5.11}$$

ϵ_n are the fermionic and ω_n the bosonic Matsubara frequencies. This is the Lindhard function: see appendix G. This has the structure, for small q/p_F and ω/qv_F:

$$\chi_0(\mathbf{q},\omega_n) \approx \nu_0\left[1 - \frac{1}{3}\left(\frac{q}{2p_F}\right)^2 - \frac{\pi}{2}\left(\frac{|\omega_n|}{qv_F}\right)\right]. \tag{5.12}$$

This leads us to the form of the bosonic propagators describing the spin interaction in the system.

Hence, to second order, we have a functional integral looking like

$$Z = \int D\phi e^{-\sum_{\mathbf{q},\omega_n}(\frac{1}{V}-\chi_0(\mathbf{q},\omega_n))\phi^2(\mathbf{q},\omega_n)}. \tag{5.13}$$

As $A \equiv V^{-1} - \chi_0$ shifts through zero the behaviour of the action will change dramatically. This change corresponds to the Stoner instability which it is believed describes the metamagnetic transition in this material. We shall use an action like this at the Stoner instability to describe the bosonic propagators in the system at criticality. In a simple minded sense, if A is positive then the perturbation expansion may be valid and the above functional integral should describe the behaviour of the system. But if A is negative then the expansion breaks down. As we choose V to be a constant evidently A is in fact a function and cannot be set to zero. It is only the leading order term which is cancelled at the Stoner instability. (This is in fact the Stoner criteria.)

The original formulation of Hertz was revisited by Millis[55] and applied to itinerant magnetic fermion systems. These are systems in which the spins are not fixed.

5.3 The Metamagnetic Model

The application of these ideas to the metamagnetic transition was later developed through several papers[56, 57, 52]. Here we present an overview of how the appropriate action is derived. This shall be the starting point for our work and that of Kim and Millis[59]. We start from a functional integral describing electrons with a spin density interaction. In general a spin density operator is given by

$$\mathbf{S}_{\mathbf{q}} = \sum_{\mathbf{k}\alpha\beta} \psi^{\dagger}_{\mathbf{k}+\mathbf{q},\alpha} \boldsymbol{\sigma}_{\alpha\beta} \psi_{\mathbf{k},\beta}. \tag{5.14}$$

With $\boldsymbol{\sigma}_{\alpha\beta}$ a vector of the Pauli matrices and α/β the spin indices. So the partition function looks like

$$Z = \int D\psi D\bar{\psi} e^{-\,\mathrm{Tr}[\bar{\psi}_{q\alpha}(\tau)(\partial_\tau + \xi_{\mathbf{q}})\psi_{q\alpha}(\tau) - J(\mathbf{q})\mathbf{S}_{\mathbf{q}}(\tau).\mathbf{S}_{-\mathbf{q}}(\tau)]}. \tag{5.15}$$

Tr is over imaginary time, momenta and spin.

Firstly however we shall briefly go through the derivation of the action for the metamagnetic quantum critical point without disorder and in the Matsubara technique. The following Hubbard-Stratonovich transformation is made:

$$\begin{aligned}\int \frac{D\boldsymbol{\phi}}{\mathcal{N}_{\boldsymbol{\phi}}} e^{\mathrm{Tr}\left[\frac{\lambda^2 \phi_{\mathbf{q}}(\tau).\phi_{-\mathbf{q}}(\tau)}{4J_{\mathbf{q}}} - i\lambda \boldsymbol{\phi}_{\mathbf{q}}(\tau).\mathbf{S}_{\mathbf{q}}(\tau)\right]} &= \int \frac{D\boldsymbol{\phi}}{\mathcal{N}_{\boldsymbol{\phi}}} e^{\mathrm{Tr}\left[\left(\frac{\lambda\phi_{\mathbf{q}}(\tau)}{2\sqrt{J_{\mathbf{q}}}} - i\sqrt{J_{\mathbf{q}}}\mathbf{S}_{\mathbf{q}}(\tau)\right).\left(\frac{\lambda\phi_{-\mathbf{q}}(\tau)}{2\sqrt{J_{\mathbf{q}}}} - i\sqrt{J_{\mathbf{q}}}\mathbf{S}_{-\mathbf{q}}(\tau)\right)\right]} \\ &\quad \times e^{\mathrm{Tr}\, J_{\mathbf{q}}[\mathbf{S}_{\mathbf{q}}(\tau).\mathbf{S}_{-\mathbf{q}}(\tau)]} \\ &= e^{\mathrm{Tr}\, J_{\mathbf{q}}[\mathbf{S}_{\mathbf{q}}(\tau).\mathbf{S}_{-\mathbf{q}}(\tau)]}. \qquad (5.16)\end{aligned}$$

Notice that this gives an interpretation of the new bosonic vector field as a spin density type of fluctuation. Integrating the now Gaussian Fermionic degrees of freedom yields

$$Z = \int D\phi e^{-S}. \qquad (5.17)$$

Where the action S is given by

$$S = \mathrm{Tr}\,\frac{\lambda^2 \boldsymbol{\phi}_{\mathbf{q}}(\tau)\boldsymbol{\phi}_{-\mathbf{q}}(\tau)}{4J} - \mathrm{Tr}\ln[G_0^{-1}(\mathbf{q},\omega_n) + i\lambda\boldsymbol{\phi}_{\mathbf{q}}(\omega_n).\boldsymbol{\sigma}]. \qquad (5.18)$$

To get to this form we use the standard result for a Gaussian fermionic functional integral and the general expression $\ln[\mathrm{Det}\,\mathbf{M}] = \mathrm{Tr}\ln[\mathbf{M}]$.

We can now expand in the bosonic field to get a Ginzburg-Landau style free energy action. To second order this will give the Lindhard function, which can be expanded.

$$S = \frac{\lambda^2}{2}\mathrm{Tr}\left[\left(\frac{1}{4J} - \frac{\nu_d}{2}\left(1 - \frac{1}{3}\frac{q^2}{4k_F^2} - \frac{\pi}{2}\frac{|\omega_n|}{qv_F}\right)\right)\boldsymbol{\phi}_{-\mathbf{q}}(-\omega_n).\boldsymbol{\phi}_{\mathbf{q}}(\omega_n)\right] \qquad (5.19)$$

is the basic action required. Note we have also assumed we are near the Stoner instability so that $J^{-1} \approx \nu_d/2$.

The full metamagnetic Landau free energy contains sixth order terms and a negative coefficient for the fourth order term. We expand around the critical point in this action to find the action we use. We can rescale the various length scales and energy scales to more convenient ones. Introducing $\xi_0 \approx k_F^{-1}$, $v \approx v_F$ and an energy scale E_0 we have, in line with the notation of Kim and Millis,

$$S = \frac{1}{2}\mathrm{Tr}[2\mathbf{h}.\boldsymbol{\phi} + \xi_0^2 q^2 \phi^2 + \frac{1}{2}\phi^4] + \mathrm{Tr}\,\frac{|\omega_n|}{vq}\phi^2. \qquad (5.20)$$

Now, the trace is defined as

$$\mathrm{Tr}(\ldots) = \frac{T}{E_0}\sum_{\omega_n}\int\frac{d^2q}{(2\pi)^2}a^2(\ldots) \tag{5.21}$$

where a is a cutoff length scale of the order of the lattice constant. E_0 is defined such that the coefficient of the fourth order term is a quarter. The following have also been rescaled:

$$h = \frac{(H - H_M)M_{sat}}{E_0}, \tag{5.22}$$

$$\phi(\mathbf{r},\tau) = \frac{M(\mathbf{r},\tau) - M^*}{M_{sat}}. \tag{5.23}$$

Where M^* is the average magnetization at the critical field; M_{sat} is the high field saturation and $H_M = E_0/M_{sat}$ is the critical field; h is the applied field measured from the transition; ϕ is the order parameter and measures the scaled magnetization of the system compared to the average; and $M(x,\tau)$ is the actual magnetization of the system.

The action in equation (5.20) is the starting point for the work of Kim and Millis. In the next chapter we shall present their phenomenological model based on this action.

The propagator for the saddle point fluctuations of action (5.20) can be shown to be

$$D(\mathbf{q},\omega_n) = \frac{1}{\frac{|\omega_n|}{vq} + \xi_0^2q^2 + h^{\frac{2}{3}}}. \tag{5.24}$$

We leave the derivation to the next chapter and confine ourselves here to mentioning models where similar Green's functions show up. Outside of metamagnetic field theories and similar electronic interactions this propagator also appears in gauge interactions. These theories are applied to the t-J model and resonant valence bond theories in high-T_c superconductors[60, 61, 62, 63, 64, 65] and to the theory of half filled Landau levels in the fractional quantum hall effect[60, 66, 67, 68]. However, due to the different nature of the coupling between these gauge fields and the electrons in these systems, we cannot directly use the same integration methods.

Chapter 6

METAMAGNETIC QUANTUM CRITICALITY

In this chapter we present a phenomenological model and calculate its outcome for scattering from stationary electronic impurities. The phenomenological model we are interested in was proposed by Kim and Millis[59]. This describes a two dimensional metamagnetic quantum critical point. Their starting point for calculations was to describe both free electrons and bosonic fluctuations and give them an interaction. The bosonic fluctuations, however, are derived by bosonizing the interacting electrons in the system. Hence it is not clear whether such a theory is physically tenable due to the apparent contradiction of both integrating out an electronic degree of freedom and simultaneously retaining it. However, it is possible to derive their action if we allow ourselves certain assumptions. We present their calculation and generalize it to the Keldysh nonequilibrium method. We also correct some mistakes which lead to an incorrect lifetime for the quasiparticles.

We are interested in calculating the self energy and the scattering integral for the model. From the self energy we find the quasiparticle lifetime, and from the scattering integral it is possible to find the resistivity of the model.

6.1 The Phenomenological Action

Kim and Millis consider a model of electrons coupled to the bosonic degree of freedom described by equation (5.20). This is an inconsistent starting point as the bosonic degree of freedom is found by integrating out the electronic degree of freedom. However, as it will be shown section 6.2, this model can be trivially derived *if* we are able to justify dividing the electrons into two "types" in an appropriate manner. This model is similar to the models mentioned at the end of the preceding chapter.

They then consider a perturbative theory of the electrons coupled to the bosonic field. To this end we first require the propagator associated with the critical fluctuations of the bosons, equation (5.20). The coupling term is given by

$$S_{\phi\psi} = g \operatorname{Tr} \bar{\psi}_\alpha(x)\sigma^z_{\alpha\beta}\psi_\beta(x)\phi(x), \tag{6.1}$$

with the usual free electron action

$$S_{el} = \operatorname{Tr} \bar{\psi}_{\mathbf{p}\alpha}(\omega_n)[-i\omega_n + \xi_{\mathbf{p}}]\psi_{\mathbf{p}\alpha}(\omega_n). \tag{6.2}$$

The coupling constant g is given by $g^2 = 4\pi^2 v_F^2/a^2 E_0 v S_F$[59]. This gives us total action to consider of

$$\begin{aligned} S &= S_{\phi\psi} + S_{el} + S_\phi \text{ and as usual} \\ Z &= \int D\phi D\psi D\bar{\psi} e^{-S}. \end{aligned} \tag{6.3}$$

The action S_ϕ is the bosonic action, derived previously, of equation (5.20):

$$S_\phi = \frac{1}{2}\operatorname{Tr}[2\mathbf{h}.\boldsymbol{\phi} + \xi_0^2 q^2\phi^2 + \frac{1}{2}\phi^4] + \frac{1}{2}\operatorname{Tr}\frac{|\omega_n|}{vq}\phi^2. \tag{6.4}$$

First we will find the propagator for the original fluctuations, described by S_ϕ. This is calculated by expanding about the saddle point

$$\left.\frac{\partial S_\phi}{\partial \phi}\right|_{\phi_0} = h + \underbrace{\hat{L}\phi_0}_{\approx 0} + \phi_0^3 = 0. \tag{6.5}$$

$\hat{L}\phi_0 = (\xi_0^2 q^2 + |\omega|/vq)\phi_0$ is ignored in a gradient expansion assuming the solution is homogeneous in space and time. Hence $\phi_0 \approx -h^{\frac{1}{3}}$. Expanding about the saddle point, $\phi \approx \phi_0 + \delta\phi$, yields

$$S_\phi = S_0 + \operatorname{Tr}\delta\phi[\hat{L} + \phi_0^2]\delta\phi. \tag{6.6}$$

This gives the bosonic propagator as

$$\begin{aligned} P(\mathbf{q},\omega_n) &= -i\langle\phi_{\mathbf{q}}(\omega_n)\phi_{\mathbf{q}}(\omega_n)\rangle \\ &= [\hat{L}+\phi_0^2]^{-1} \\ &= \frac{1}{\xi_0^2 q^2 + \frac{|\omega_n|}{vq} + h^{\frac{2}{3}}}, \end{aligned} \tag{6.7}$$

$P(\mathbf{q},\omega_n)$ describes the bosonic degree of freedom about its saddle point solution. This is used in perturbative field theory calculations coupled to free electrons.

6.2 Derivation of the Phenomenological Action

We wish to derive a model which describes electronic and bosonic degrees of freedom interacting by a simple coupling term (see equation (6.1).) Our aim in this section is not to justify the model of Kim and Millis presented in section 6.1, but to elucidate the necessary assumptions that are required for it to be valid. If we assume we can divide the electrons of our system into two categories then it is possible to derive an action like that of Kim and Millis, given a few further assumptions. We will bosonize and integrate out one electron degree of freedom and leave the other remaining. We shall leave a discussion of the possible ways of justifying this move to chapter 7. As with the previous derivations for metamagnetic actions we start with a spin density interaction of the form

$$\hat{H}_i = -\frac{1}{2}\operatorname{Tr} J_{\mathbf{q}}\hat{S}_{\mathbf{q}}\hat{S}_{\mathbf{q}} \text{ where } \hat{S}_{\mathbf{q}} = \sum_{\mathbf{k},\alpha,\beta}\hat{\psi}^{\dagger}_{\mathbf{k}+\mathbf{q},\alpha}\sigma^z_{\alpha\beta}\hat{\psi}_{\mathbf{k},\beta}. \tag{6.8}$$

Then we assume that we can divide the electrons into two types: $\psi^A(x)$ and $\psi^B(x)$. It is not clear, however, what would distinguish between them. The A electrons are the current carrying electrons in the 2-d planes. Most importantly the electrons must obey the property $\langle\bar{\psi}^A\psi^B\rangle = \langle\bar{\psi}^B\psi^A\rangle = 0$. For example, we may wish to distinguish between fast and slow electrons or those in the planes and those along the c-axis. Thus, from equation(5.15), we have

$$
\begin{aligned}
Z = \int D\psi^A D\bar{\psi}^A D\psi^B D\bar{\psi}^B e^{-\int d\tau \sum_{\mathbf{q},\alpha}[\bar{\psi}^A_{\mathbf{q}\alpha}(\tau)(\partial_\tau+\xi_{\mathbf{q}})\psi^A_{\mathbf{q}\alpha}(\tau)+\bar{\psi}^B_{\mathbf{q}\alpha}(\tau)(\partial_\tau+\xi_{\mathbf{q}})\psi^B_{\mathbf{q}\alpha}(\tau)]} \\
e^{-\int d\tau \sum_{\mathbf{q}} J S^B_{\mathbf{q}}(\tau) S^B_{\mathbf{q}}(\tau) + 2J\int d\tau \int d^2\mathbf{r} \sum_\alpha [\bar{\psi}^B_\alpha \bar{\psi}^A_{\bar{\alpha}} \psi^A_\alpha \psi^B_{\bar{\alpha}} - \bar{\psi}^B_\alpha \bar{\psi}^A_{\bar{\alpha}} \psi^A_{\bar{\alpha}} \psi^B_\alpha]}.
\end{aligned}
\tag{6.9}
$$

The interaction of the A-electrons with themselves has been neglected. We consider them to be free electrons in the a-b plane.

We can now perform a Hubbard-Stratonovich transform on the B-electron interaction term. This introduces a field ϕ and the Fermionic integral over the B-electrons becomes Gaussian. The partition function then becomes

$$
\begin{aligned}
Z = \int D\phi e^{\int d\tau d^2\mathbf{r} \frac{\phi^2(x)\lambda^2}{4J}} \int D\psi^A D\bar{\psi}^A e^{-\int d\tau d^2\mathbf{r} \sum_\alpha \bar{\psi}^A_\alpha(x)(\partial_\tau+\hat{\xi}_{\mathbf{r}})\psi^A_\alpha(x)} \\
\int D\psi^B D\bar{\psi}^B e^{-\int d\tau d^2\mathbf{r} \bar{\psi}^B_\alpha(x)[(\partial_\tau+\hat{\xi}_{\mathbf{r}})\delta_{\alpha\beta}+i\lambda\phi(x)\sigma^z_{\alpha\beta}]\psi^B_\beta(x)} \\
\underbrace{e^{-2J\int d\tau d^2\mathbf{r} \sum_{\alpha\beta} \bar{\psi}^B_\alpha [\sum_\delta \bar{\psi}^A_\delta \psi^A_\delta \delta_{\alpha\beta}\sigma^x_{\delta\alpha} - \bar{\psi}^A_\beta \psi^A_\alpha \sigma^x_{\alpha\beta}]\psi^B_\beta}}_{\equiv e^{-2J\,\mathrm{Tr}\,\bar{\psi}^B_\alpha(x) A_{\alpha,\beta}(x)\psi^B_\beta(x)}}
\end{aligned}
\tag{6.10}
$$

which we will integrate over $D\psi^B$. (Arguments are dropped for clarity where it is unambiguous what is intended.) After performing this integral we shall have

$$
\begin{aligned}
Z = \int D\phi e^{\int d\tau d^2\mathbf{r} \frac{\phi^2(x)\lambda^2}{4J}} \int D\psi^A D\bar{\psi}^A e^{-\int d\tau d^2\mathbf{r} \sum_\alpha \bar{\psi}^A_\alpha(x)(\partial_\tau+\hat{\xi}_{\mathbf{r}})\psi^A_\alpha(x)} \\
e^{\mathrm{Tr}\ln G_0^{-1}(\mathbf{q},\omega_n)} e^{\mathrm{Tr}\ln[1+G_0(\mathbf{q},\omega_n)i\lambda\phi(\mathbf{q},\omega_n)\sigma^z_{\alpha\beta}+G_0(\mathbf{q},\omega_n)2JA_{\alpha,\beta}(\mathbf{q},\omega_n)]}.
\end{aligned}
\tag{6.11}
$$

We shall Taylor expand the terms

$$
e^{\mathrm{Tr}\ln G_0^{-1}} e^{\mathrm{Tr}\ln[1+G_0 i\lambda\phi\sigma^z_{\alpha\beta}+G_0 2JA(\alpha,\beta)]}. \tag{6.12}
$$

The small parameters for this expansion are $J/\varepsilon_F \ll 1$ and $\lambda/\varepsilon_F \ll 1$. To leading order we find the terms describing ϕ, as in the normal metamagnetic action equation (5.20), and also the interaction term:

$$
e^{-4J\lambda\,\mathrm{Tr}\,G_0(x-x')\phi(x)G_0(x'-x)S^A(x')} = e^{-4J\lambda\,\mathrm{Tr}\,\Pi(\mathbf{q},\omega_n)\phi(\mathbf{q},\omega_n)S^A(\mathbf{q},\omega_n)}. \tag{6.13}
$$

Where $\Pi(\mathbf{q}, \omega_n)$ is the Lindhard function, see appendix G. To lowest order this gives a constant and hence

$$S_{\phi\psi} = (4J\lambda\omega_0)\,\mathrm{Tr}\,\bar{\psi}^A_\alpha(x)\sigma^z_{\alpha\beta}\psi^A_\beta(x)\phi(x). \tag{6.14}$$

Together with the bosonic action over ϕ and the free electron action for ψ^A this reproduces the action of Kim and Millis, equation (6.3). To summarize we require the following to be true: $\langle\bar{\psi}^A\psi^B\rangle = \langle\bar{\psi}^B\psi^A\rangle = 0$; $J/\varepsilon_F \ll 1$ and $\lambda/\varepsilon_F \ll 1$. J measures the coupling between the A and B electrons and λ measures the strength of the interaction between B electrons.

6.3 Keldysh Formulation

We now present the metamagnetic model in the Keldysh representation. We can write the original action, equation (5.15), on the Keldysh contour as

$$iS_1 = i\,\mathrm{Tr}\,\sigma^3_{ij}[\bar{\psi}_j(\mathbf{q},\omega)(\hat{\epsilon} - \hat{\xi}_{\mathbf{q}})\psi_j(\mathbf{q},\omega) - J_{\mathbf{q}}(\mathbf{S}_{\mathbf{q}}(\omega).\mathbf{S}_{-\mathbf{q}})_j(\omega)]. \tag{6.15}$$

Where the trace now includes a trace over the indices i, j labelling the upper and lower time contour. We can now perform a Hubbard-Stratonovich transformation on the quartic part of this action. This will introduce a bosonic field to integrate over and we will be left with

$$\begin{aligned} iS_2 &= i\,\mathrm{Tr}\left[\frac{\lambda^2}{4J_{\mathbf{q}}}\hat{\mathbf{\Phi}}^T_{\mathbf{q},\alpha}(\omega)\boldsymbol{\sigma}^3\hat{\mathbf{\Phi}}_{-\mathbf{q},\alpha}(-\omega)\right] + i\,\mathrm{Tr}[-i\bar{\psi}_{i,\mathbf{q},\alpha}(\omega)\lambda\hat{\boldsymbol{\phi}}^i_{\mathbf{q},\alpha}(\omega).\sigma_{\alpha\beta}\hat{\boldsymbol{\gamma}}^i\psi_{i,\mathbf{q},\beta}(\omega)] + iS_0 \\ iS_0 &= i\,\mathrm{Tr}\,\sigma^3_{ij}\bar{\psi}_{i,\mathbf{q},\alpha}(\omega)[\hat{\epsilon} + \xi_{\mathbf{q}}]\psi_{i,\mathbf{q},\alpha}(\omega) \\ Z &= \int D\phi D\psi D\bar{\psi}e^{iS_2}. \end{aligned} \tag{6.16}$$

With the matrices

$$\hat{\boldsymbol{\gamma}}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \hat{\boldsymbol{\gamma}}^2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.17}$$

which are also traced over. Note that the σ^3 matrix refers to the Keldysh space as does the i index, all others refer to the spin structure. Also $\hat{\mathbf{\Phi}}^T = (\hat{\phi}_1, \hat{\phi}_2)$ is in Keldysh space.

We now rotate to the standard Larkin-Ovchinnikov basis[43] using:

$$\mathbf{G} = \mathbf{L}_0 \boldsymbol{\sigma}^3 \hat{\mathbf{G}} \mathbf{L}_0^\dagger \qquad \phi^{1/2} = \frac{1}{2}(\hat{\phi}^1 \pm \hat{\phi}^2). \tag{6.18}$$

The $\mathbf{L}_0$ matrices are defined in section 2.3. The vertex matrices are now given by equation (2.47). After performing this rotation we integrating out the fermionic degrees of freedom which leaves us with the action

$$iS = i\,\mathrm{Tr}[\mathbf{\Phi}^T_{\mathbf{q},\alpha}(\omega)\sigma^1\mathbf{\Phi}_{-\mathbf{q},\alpha}(-\omega)\lambda^2/4J] + \mathrm{Tr}\ln[G_0^{-1}(\mathbf{q},\omega) - i\lambda\phi^\delta_{\mathbf{q},\alpha}(\omega).\boldsymbol{\sigma}_{\alpha\beta}\gamma^\delta]. \tag{6.19}$$

Expanding the logarithm to the first non-vanishing order leaves

$$iS = i\frac{\lambda^2}{2}\,\mathrm{Tr}[\mathbf{\Phi}^T_{\mathbf{q}}(\omega)\sigma^1(\underbrace{1/2J + \mathbf{\Pi}(\mathbf{q},\omega)}_{\equiv\hat{\mathbf{L}}(\mathbf{q},\omega)})\mathbf{\Phi}_{\mathbf{q}}(\omega)] \tag{6.20}$$

for the action. (We are only interested in the z-component of the spin). $\mathbf{\Pi}(\mathbf{q},\omega)$ is the Lindhard operator as before.

We require the terms in the action corresponding to the magnetic field and the quartic term. We introduce them in the rotation where $\hat{\phi}^{1,2}$ are the Bosonic fields on the upper/lower part of the Keldysh contour. Noting that $h(\hat{\phi}_1 - \hat{\phi}_2) = h\phi_2$ and $1/4[\hat{\phi}_1^4 - \hat{\phi}_2^4] = \phi_1\phi_2(\phi_1^2 + \phi_2^2)$ we find

$$iS = i\,\mathrm{Tr}[h\phi_2\lambda + (\lambda^2/2)\Phi^T\sigma_1\hat{L}\Phi + \lambda^4\phi_1\phi_2(\phi_1^2+\phi_2^2)]. \tag{6.21}$$

If we expand around the saddle point of this action we can find the propagator we require:

$$\mathbf{P}(\mathbf{q},\omega) = \begin{pmatrix} (-i\omega/vq + \xi_0^2q^2 + h^{2/3})^{-1} & \coth(\omega/2T)[P^R - P^A] \\ 0 & (i\omega/vq + \xi_0^2q^2 + h^{2/3})^{-1} \end{pmatrix}, \tag{6.22}$$

which can then be used in the usual Keldysh diagram technique.

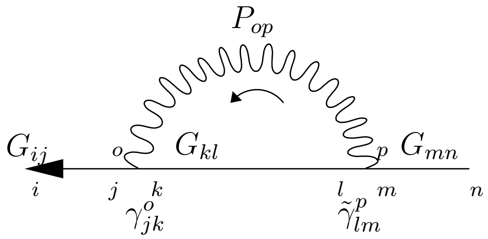


Figure 6.1: The lowest order self energy diagram.

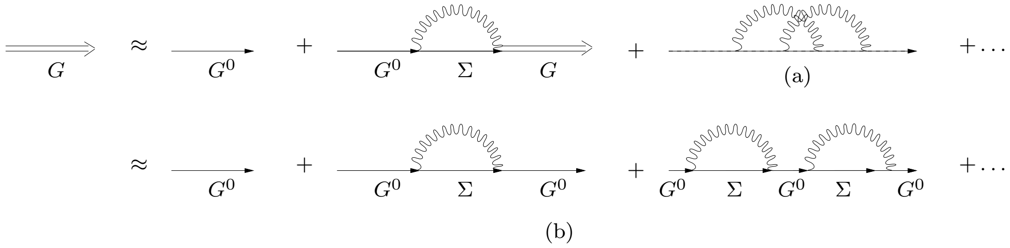


Figure 6.2: Some of the self energy diagrams for the Green's function. We consider the diagrams included in (b). Contributions like (a) are neglected.

6.4 The Self Energy

In this section we present a calculation of the lowest order self energy contributions to the electron Green's functions due to the bosonic field. We demonstrate how to recover these results in the Keldysh formalism and also where the previous derivation of the lifetime erred[59]. We find that, away from the quantum critical point, the lifetime of the quasiparticles is similar to that of a Fermi-liquid, modified by a logarithmic term. There is always a Fermi-liquid like contribution to the lifetime but it is never dominant in the cases we consider.

We want the lifetime of the particles interacting with the bosonic field. We will consider the self energy which appears in Dyson's equation: $G^{-1} = G_0^{-1} - \Sigma$. In the Green's function,

$$G(\mathbf{p}, \varepsilon) = -\frac{i}{Z} \langle \mathrm{T}\, \hat{S} \hat{\psi}^\dagger_{\mathbf{p}}(\varepsilon) \hat{\psi}_{\mathbf{p}}(\varepsilon) \rangle, \tag{6.23}$$

we expand the S-matrix to find the corrections. The S-matrix is given by

$$\mathrm{T}\, \hat{S} = e^{-ig \int_{-\infty}^{\infty} dt\, \mathrm{Tr}[\hat{\psi}^\dagger_\alpha(x) \sigma^z_{\alpha\beta} \phi(x) \hat{\psi}_\beta(x)]}. \tag{6.24}$$

Expanding this we obtain the first order correction. This is given by

$$G(\mathbf{p},\varepsilon) = G^0(\mathbf{p},\varepsilon) + G^0(\mathbf{p},\varepsilon)\underbrace{i^2g^2\int\frac{d^2q}{4\pi^2}\frac{d\omega}{2\pi}G^0(\mathbf{p}-\mathbf{q},\varepsilon-\omega)\langle\phi_{\mathbf{q}}(\omega)\phi_{\mathbf{q}}(\omega)\rangle}_{\equiv\Sigma_1}G^0(\mathbf{p},\varepsilon)$$
$$+\cdots. \tag{6.25}$$

Summing all such non-crossed contributions, see section 2.2, will give us the required Green's function. Thus we neglect all contributions containing crossed bosonic propagators and other higher order terms. Using $P(\mathbf{q},\omega) = -i\langle\phi_q\phi_q\rangle$ we can write the self energy at zero temperature as

$$\Sigma_1(\mathbf{p},t) = -ig^2a^2E_0\int\frac{d^2\mathbf{q}}{(2\pi)^2}\int\frac{d\omega}{2\pi}G^0(\mathbf{p}-\mathbf{q},\varepsilon-\omega)P(\mathbf{p},\varepsilon). \tag{6.26}$$

This is the self energy that Kim and Millis deal with.

This can also be formulated in the Keldysh technique. The self energy in Keldysh becomes:

$$\Sigma_{ij}(\mathbf{p},\varepsilon) = -i(g^2a^2E_0)\int\frac{d^2q}{(2\pi)^2}\frac{d\omega}{2\pi}\gamma^m_{ik}G^0_{kl}(\mathbf{p}-\mathbf{q},\varepsilon-\omega)P_{mp}(\mathbf{q},\omega)\tilde{\gamma}^p_{lj}. \tag{6.27}$$

Hence we can write, in equilibrium,

$$\begin{aligned}
\Sigma^R(\mathbf{p},\varepsilon) &= -\frac{ig^2a^2E_0}{2}\int\frac{d^2q}{(2\pi)^2}\frac{d\omega}{2\pi}[G^R(\mathbf{p}-\mathbf{q},\varepsilon-\omega)P^K(\mathbf{q},\omega)\\
&\qquad +G^K(\mathbf{p}-\mathbf{q},\varepsilon-\omega)P^A(\mathbf{q},\omega)]\\
\Sigma^A(\mathbf{p},\varepsilon) &= [\Sigma^R(\mathbf{p},\varepsilon)]^*\\
\Sigma_{21}(\mathbf{p},\varepsilon) &= 0\\
\Sigma^K(\mathbf{p},\varepsilon) &= -\frac{ig^2a^2E_0}{2}\int\frac{d^2q}{(2\pi)^2}\frac{d\omega}{2\pi}[1+h(\varepsilon-\omega)\tilde{h}(\omega)]\\
&\qquad (G^R(\mathbf{p}-\mathbf{q},\varepsilon-\omega)-G^A(\mathbf{p}-\mathbf{q},\varepsilon-\omega))(P^R(\mathbf{q},\omega)-P^A(\mathbf{q},\omega))
\end{aligned} \tag{6.28}$$

$\tilde{h}(\omega)$ is the bosonic distribution function. The usual rules for Keldysh matrices hold. We will consider zero temperature, therefore we can write

$$\Sigma^R(\mathbf{p},\varepsilon) = -\frac{iv_F^2}{8\pi^2 vp_F}\int d^2q d\omega \big[\operatorname{sgn}(\omega)G^R(\mathbf{p}-\mathbf{q},\varepsilon-\omega)(P^R(\mathbf{q},\omega)-P^A(\mathbf{q},\omega)) \\ +\operatorname{sgn}(\varepsilon-\omega)(G^R(\mathbf{p}-\mathbf{q},\varepsilon-\omega)-G^A(\mathbf{p}-\mathbf{q},\varepsilon-\omega))P^A(\mathbf{q},\omega)\big]. \tag{6.29}$$

After performing the frequency integration it is the possible to see that we can write this in terms of the zero temperature version (as we would expect):

$$i\Im\Sigma^R(\mathbf{p},\varepsilon) = i\Im\Sigma_1(\mathbf{p},|\varepsilon|). \tag{6.30}$$

This shall be calculated presently. In order to facilitate comparisons with the calculation of Kim and Millis we will present the zero temperature calculation and then recap the results for the Keldysh form.

From equation (6.26) the self energy is

$$\Sigma_1(\mathbf{p},\varepsilon) = \frac{v_F^2}{4\pi^2 vp_F}\int d^2q\int_{-\infty}^{\infty}\frac{d\omega}{\varepsilon+\omega-\xi_{\mathbf{p}+\mathbf{q}}+i\delta\operatorname{sgn}(\xi_{\mathbf{p}+\mathbf{q}})}\frac{vq}{|\omega|+ia_q}, \tag{6.31}$$

where $a_q = vq(\xi_0^2 q^2 + h^{\frac{2}{3}})$. We have used the explicit form for the coupling constant $g^2 = 4\pi^2 v_F^2/a^2 E_0 v S_F$, where S_F is the length of the Fermi surface in momentum space. We will also define $b_q = a_q/v_F q$.

Let us define $B = \xi - \varepsilon$ and $B^+ = \xi - \varepsilon - i\delta\operatorname{sgn}(\xi)$. Then we may write

$$\Sigma_1(\mathbf{p},\varepsilon) = \frac{v_F^2}{4\pi^2 p_F}\int d^2q q\left\{\lim_{z\to -B^+} - \lim_{z\to B^+}\right\}\underbrace{\int_0^{\infty} d\omega\frac{1}{\omega+ia_q}\frac{1}{\omega+z}}_{=\frac{1}{z-ia_q}\ln[\frac{z}{ia_q}]}. \tag{6.32}$$

From the analytic structure of the integral over ω we see that the branch cut of the logarithm is along the negative real axis for z. Taking the imaginary part of the self energy only we have

$$i\Im\Sigma_1(\mathbf{p},\varepsilon) = \frac{iv_F}{2\pi p_F\xi_0^2}\int_0^{\infty} dxx\int_{-\pi}^{\pi} d\theta\,\Theta[(\epsilon_x-\cos(\theta))\cos(\theta)]\frac{\cos(\theta)-\epsilon_x}{(\cos(\theta)-\epsilon_x)^2+b_x^2}. \tag{6.33}$$

We have rescaled $x = \xi_0 q$, written $\epsilon_x = \varepsilon\xi_0/v_F x$ and linearized $\xi_{\mathbf{p}+\mathbf{q}} \approx v_F\mathbf{p}.\mathbf{q}/|\mathbf{q}|$. We note that when we consider the scattering integral this linearization is invalid. We will demonstrate later that there are no such problems with the self energy and the standard linearization procedure works. The real part is

$$\Re\Sigma_1(\mathbf{p},\varepsilon) = \frac{v_F}{4\pi^2 p_F \xi_0^2}\int_0^\infty dx x\int_{-\pi}^{\pi} d\theta \frac{2(\cos(\theta)-\epsilon_x)\ln[b_x/(\cos(\theta)-\epsilon_x)]}{(\cos(\theta)-\epsilon_x)^2+b_x^2} \\ -\frac{b_x\pi\,\mathrm{sgn}[\cos(\theta)]}{(\cos(\theta)-\epsilon_x)^2+b_x^2}. \tag{6.34}$$

To calculate the imaginary part we first split the x integral into the two regions where $\epsilon_x \gg 1$ and $\epsilon_x \ll 1$. Firstly the low momenta region can be checked. As $\varepsilon \ll \epsilon_F$ we can also take $b_x \ll 1$ in this region and hence we have

$$\begin{aligned}\Im\Sigma_1((\mathbf{p},\varepsilon)|_{\epsilon_x\gg 1} &\approx \frac{v_F}{2\pi p_F \xi_0^2}\int_0^{\frac{|\varepsilon|\xi_0}{v_F}} dx x\int_{-\pi}^{\pi} d\theta\,\Theta[(\epsilon_x-\cos(\theta))\cos(\theta)]\frac{\cos(\theta)-\epsilon_x}{(\cos(\theta)-\epsilon_x)^2+b_x^2} \\ &\approx -\frac{\varepsilon}{p_F\xi_0}\int_0^{\frac{|\varepsilon|\xi_0}{v_F}} dx\frac{1}{\epsilon_x^2}\left(1-\frac{b_x^2}{\epsilon_x^2}\right)+\cdots \\ &\approx -\frac{1}{6}\,\mathrm{sgn}(\varepsilon)\frac{\varepsilon^2}{\epsilon_F}+\cdots .\end{aligned} \tag{6.35}$$

Higher order terms will bring in dependence on the applied field h, but they are neglected here as we are interested in $h \ll 1$. This result is the same as the inverse lifetime you find for single particle excitations in a Fermi liquid, $\tau^{-1} \sim \varepsilon^2/\epsilon_F$, which can be found from Fermi's golden rule. However the remaining contributions from momenta between $\frac{|\varepsilon|}{v_F}$ and $\xi_0^{-1} \approx p_F$ give more important corrections.

It is more advisable to remove the theta function by explicitly rewriting the limits of the angular integral. We can write

$$\Im\Sigma_1((\mathbf{p},\varepsilon) = \frac{v_F\,\mathrm{sgn}(\varepsilon)}{\pi p_F\xi_0^2}\left[\int_0^{\frac{|\varepsilon|\xi_0}{v_F}} dx x\int_0^{\pi/2} d\theta\frac{\cos(\theta)-|\epsilon_x|}{(\cos(\theta)-|\epsilon_x|)^2+b_x^2}+ \right. \\ \left. \int_{\frac{|\varepsilon|\xi_0}{v_F}}^\infty dx x\int_{\cos^{-1}(|\epsilon_x|)}^{\pi/2} d\theta\frac{\cos(\theta)-|\epsilon_x|}{(\cos(\theta)-|\epsilon_x|)^2+b_x^2}\right]. \tag{6.36}$$

This expression is exact except for the linearization of ξ in the electron Green's function.

We will also substitute $y = \cos(\theta) - |\epsilon_x|$ to yield

$$\Im\Sigma_1((\mathbf{p},\varepsilon) = \frac{v_F \operatorname{sgn}(\varepsilon)}{\pi p_F \xi_0^2}\Bigg[\int_0^{\frac{|\varepsilon|\xi_0}{v_F}} dxx \int_{-|\epsilon_x|}^{1-|\epsilon_x|} dy \frac{y}{y^2+b_x^2}\frac{1}{\surd(1-[y+|\epsilon_x|]^2)} + \int_{\frac{|\varepsilon|\xi_0}{v_F}}^{\infty} dxx \int_{-|\epsilon_x|}^{0} dy \frac{y}{y^2+b_x^2}\frac{1}{\surd(1-[y+|\epsilon_x|]^2)}\Bigg]. \tag{6.37}$$

The first of these terms will again give the Fermi liquid result $\tau^{-1} \sim \varepsilon^2/\epsilon_F$. However, the second term also needs to be taken care of. The integral over x restricts ϵ_x to lie between 0 and 1 and y is bounded between $-|\epsilon_x|$ and 0. We shall approximate the square root as small ($\ll 1$) x and y are the most important terms:

$$\frac{1}{\sqrt{1-[y+\epsilon_x]^2}} \approx 1. \tag{6.38}$$

Now

$$\Im\Sigma_1(\mathbf{p},\varepsilon) = -\frac{\operatorname{sgn}(\varepsilon)\varepsilon^2}{6\epsilon_F} - \frac{v_F \operatorname{sgn}(\varepsilon)}{2\pi p_F \xi_0^2}\int_{\frac{|\varepsilon|\xi_0}{v_F}}^{\infty} dxx \ln\left[\frac{(\varepsilon\xi_0/v)^2 + (x^3+xh^{2/3})^2}{(x^3+xh^{2/3})^2}\right]. \tag{6.39}$$

This integral over x can be calculated in the same limits as Kim and Millis used giving

$$\begin{aligned}
\Im\Sigma_1(\mathbf{p},\varepsilon) &\to -\operatorname{sgn}(\varepsilon)\left(\frac{\varepsilon}{h^{2/3}}\right)^2 \frac{1}{\pi\varepsilon_F}\ln\left[\frac{\varepsilon_F}{|\varepsilon|}\right] \text{ as } \varepsilon \to 0 \text{ at } h \neq 0 && (6.40)\\
\Im\Sigma_1(\mathbf{p},\varepsilon) &\to -\operatorname{sgn}(\varepsilon)|\varepsilon|^{2/3}\varepsilon_F^{1/3} C \text{ as } h \to 0 \text{ at } \varepsilon \neq 0. && (6.41)\\
C &= \left(\frac{v_F}{v}\right)^{\frac{2}{3}} \frac{1}{\pi 2^{\frac{4}{3}}}\frac{1}{(\xi_0 p_F)^{\frac{4}{3}}} \underbrace{\int_0^{\infty} dxx \ln[1+x^{-6}]}_{=\frac{\pi}{\sqrt{3}}} && (6.42)
\end{aligned}$$

In the limit $h \to 0$ it gives the desired result, but in the limit $\varepsilon \to 0$ it differs from what Kim and Millis found. In both these limits the Fermi liquid like term can be neglected. Note in the case $h \to 0$ we extend the lower limit of integration down to zero and subtract the excess. This excess just gives higher order contributions which are neglected and the remaining integral can be calculated.

We can also check that the quadratic q^2 terms in the dispersion $\xi_{\mathbf{q}}$ do not contribute to the self energy, within the appropriate approximations. This is necessary as we find that in the scattering integral we must include these terms. If we include them and repeat the above calculation we will have

$$\begin{aligned}\Im\Sigma_1(\mathbf{p},\varepsilon) = -\frac{v_F\,\mathrm{sgn}(\varepsilon)}{\pi p_F \xi_0^2}\Bigg\{ &\int_0^{\frac{|\varepsilon|\xi_0}{v}} dx x \int_{1-\epsilon_x+\frac{x}{2}}^{-\epsilon_x} dy \frac{y}{y^2+b^2}\frac{1}{\sqrt{1-(y+\epsilon_x-x/2)^2}} \\ &+ \int_{\frac{|\varepsilon|\xi_0}{v}}^{2} dx x \int_0^{-\epsilon_x} dy \frac{y}{y^2+b^2}\frac{1}{\sqrt{1-(y+\epsilon_x-x/2)^2}} \\ &+ \int_2^{1+\sqrt{1+\frac{2|\varepsilon|\xi_0}{v}}} dx x \int_0^{\frac{x}{2}-1-\epsilon_x} dy \frac{y}{y^2+b^2}\frac{1}{\sqrt{1-(y+\epsilon_x-x/2)^2}}\Bigg\}. \end{aligned} \tag{6.43}$$

Large momentum terms can be shown not to contribute as expected and in these regions the additional terms can safely be ignored as it should be small x and y which contribute the most important terms.

From equations (6.40) and (6.41) we find the Keldysh self energy contributions. In the limit $\varepsilon \to 0$ and with $h \neq 0$ we have

$$\Im\mathbf{\Sigma}(\mathbf{p},\varepsilon) \to \begin{pmatrix} -\left(\frac{\varepsilon}{h^{2/3}}\right)^2 \frac{1}{\pi\varepsilon_F}\ln\left[\frac{\varepsilon_F}{|\varepsilon|}\right] & \tilde{h}(\varepsilon)2\Im\Sigma^R(\mathbf{p},\varepsilon) \\ 0 & \left(\frac{\varepsilon}{h^{2/3}}\right)^2 \frac{1}{\pi\varepsilon_F}\ln\left[\frac{\varepsilon_F}{|\varepsilon|}\right] \end{pmatrix}. \tag{6.44}$$

In the limit $h \to 0$ with $\varepsilon \neq 0$ we have

$$\Im\mathbf{\Sigma}(\mathbf{p},\varepsilon) \to \begin{pmatrix} -|\varepsilon|^{2/3}\varepsilon_F^{1/3}C & \tilde{h}(\varepsilon)2\Im\Sigma^R(\mathbf{p},\varepsilon) \\ 0 & |\varepsilon|^{2/3}\varepsilon_F^{1/3}C \end{pmatrix}. \tag{6.45}$$

$\tilde{h}(\varepsilon)$ is the bosonic distribution function as defined previously. We now address the differences between our result derived above and the result of Kim and Millis. We believe there is a simple flaw in their work which invalidates one limit of the lifetime. The limit $h \to 0$ remains functionally unchanged. This standard result is robust to most variations of the self energy and is caused by the form of the bosonic propagator.

To reproduce the results of Kim and Millis we will use a different method. Firstly we use

$$P(\mathbf{q},\omega) = \frac{1}{\xi_0^2 q^2 + |\omega|/vq + h^{2/3}} \tag{6.46}$$

for the bosonic propagator instead of

$$P(\mathbf{q},\omega) = \frac{1}{\xi_0^2 q^2 - i|\omega|/vq + h^{2/3}}. \tag{6.47}$$

It is simple to check that this is not the correct zero temperature form by comparison with the result in the Keldysh formulation or by looking at the actual zero temperature derivation of the propagator. Using the wrong propagator will not change the result for the limits we are interested in (the limits $\varepsilon \to 0$ and $h \to 0$ are robust to this trivial change) but will alter the numerical coefficients of the results. However, for the following calculation we will work with the correct zero temperature form for the propagator. This will highlight the differences with our method. The derivation with the alternative propagator works in an identical way and we shall quote it at the end.

We will split the frequency integral of equation (6.31) into two regions $0 \to \varepsilon$ and $\varepsilon \to \infty$. First let us examine the region $\varepsilon \to \infty$, which we shall label Σ'. If we use the approximation $\omega \gg \varepsilon$ in this region we can write

$$\Sigma'(\mathbf{p},\varepsilon) = \frac{v_F^2}{4\pi^2 p_F} \int_0^\infty dq q^2 \int_{-\pi}^{\pi} d\theta \int_\varepsilon^\infty \frac{d\omega}{\omega + ia}\Bigg[\frac{1}{\omega - v_F q\cos(\theta) + i\delta} - \frac{1}{\omega + v_F q\cos(\theta) + i\delta}\Bigg]. \tag{6.48}$$

The validity of this approximation and a similar one used presently is an important issue here and we shall discuss it at the end of this section. We are interested only in the imaginary part, so after frequency integration we can rearrange this to give

$$\Sigma'(\mathbf{p},\varepsilon) = \frac{iv_F}{4\pi p_F}\int_0^\infty dq q \int_{-\pi}^{\pi} d\theta \frac{\cos(\theta)}{\cos^2(\theta) + b_q^2}\big[\,\Theta[\cos(\theta) + \epsilon_q] - \Theta[\cos(\theta) - \epsilon_q]\big]. \tag{6.49}$$

Splitting the q integral into regions where $\epsilon_q > 1$ and $\epsilon_q < 1$ we can perform the angular integration:

$$\Sigma'(\mathbf{p},\varepsilon) = \frac{iv_F}{4\pi p_F}\int_{\frac{|\varepsilon|}{v_F}}^\infty dq q \ln\left[\frac{\epsilon_q^2 + b_q^2}{b_q^2}\right]. \tag{6.50}$$

In the limit $\varepsilon \to 0$ this gives terms of higher than linear order. Hence it shall be neglected in comparison with the terms from the remaining frequency region.

Now for the other region. We will make the approximation $\varepsilon \gg \omega$ and write

$$\begin{aligned}
\Sigma_1(\mathbf{p},\varepsilon) &\approx \frac{v_F^2}{4\pi^2 p_F}\int_0^\infty dq q^2 \int_{-\pi}^{\pi} d\theta \int_0^{\varepsilon} d\omega \frac{1}{\varepsilon - v_F q\cos(\theta) + i\delta\,\mathrm{sgn}(\varepsilon)}\frac{1}{|\omega| + i a_q} \\
&= -\frac{i v_F^2}{2\pi^2 p_F}\int_0^\infty dq q^2 \underbrace{\int_{-\pi}^{\pi} d\theta \frac{1}{\varepsilon - v_F q\cos(\theta) + i\delta\,\mathrm{sgn}(\varepsilon)}}_{=\frac{2\pi}{v_F q}\frac{\mathrm{sgn}(\varepsilon)}{\sqrt{1-(\varepsilon/v_F q)^2}}} \tan^{-1}\left(\frac{|\varepsilon|}{a_q}\right) \\
&\approx -\frac{i v_F\,\mathrm{sgn}(\varepsilon)}{\pi p_F \xi_0^2}\int_0^\infty dx x\left[\tan^{-1}\left(\frac{|\varepsilon|\xi_0/v}{x^3 + x h^{\frac{2}{3}}}\right)\right].
\end{aligned} \tag{6.51}$$

Again we have neglected the real part. In the limit $\varepsilon \to 0$ we find a linear ε dependence, and not the dependence we found previously in equation (6.40). The necessary limits leave us with:

$$\Im\Sigma_1(\mathbf{p},\varepsilon) = -i\frac{\varepsilon}{h^{\frac{1}{3}}}\frac{v_F}{2 v p_F \xi_0} \text{ as } \varepsilon \to 0 \text{ at } h \neq 0 \text{ and} \tag{6.52}$$

$$\Im\Sigma_1(\mathbf{p},\varepsilon) = -i\,\mathrm{sgn}(\varepsilon)|\varepsilon|^{\frac{2}{3}}\frac{v_F}{2 v^{\frac{2}{3}} p_F \xi_0^{\frac{4}{3}}} \text{ as } h \to 0 \text{ at } \varepsilon \neq 0 \tag{6.53}$$

for the self energy.

If we perform the same calculation with the incorrect propagator we will find

$$\Sigma_1(\mathbf{p},\varepsilon) = -\frac{i v_F}{\pi p_F \xi_0^2}\int_0^\infty dx x \ln\left[\frac{\varepsilon\xi_0/v + x^3 + x h^{\frac{2}{3}}}{x^3 + x h^{\frac{2}{3}}}\right] \tag{6.54}$$

exactly as they have. (The calculation works in an identical way). This also has linear ε dependence in the limit $\varepsilon \to 0$. The required limits result in:

$$\Sigma_1(\mathbf{p},\varepsilon) \to -\frac{i\varepsilon}{h^{1/3}}\frac{v_F}{\pi v p_F \xi_0} \text{ as } \varepsilon \to 0 \text{ at } h \neq 0 \text{ and} \tag{6.55}$$

$$\Sigma_1(\mathbf{p},\varepsilon) \to i\,\mathrm{sgn}(\varepsilon)|\varepsilon|^{2/3}\frac{v_F}{\sqrt{3} v^{\frac{2}{3}} p_F \xi_0^{\frac{4}{3}}} \text{ as } h \to 0 \text{ at } \varepsilon \neq 0. \tag{6.56}$$

As can be seen, these do not differ from using the correct propagator in any significant way.

The problem with these calculations is the assumption that the region around ε is unimportant in the frequency integral. However it is clear from the form of the electron Green's function, $[\varepsilon + \omega - \xi_{\mathbf{p}+\mathbf{q}} + i\delta\,\mathrm{sgn}(\xi)]^{-1}$, that this is untrue. The problem is in fact

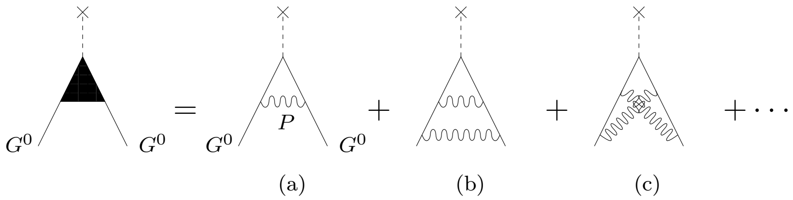


Figure 6.3: The scattering vertex correction. We include contributions like diagrams (a) and (b). (c) and other crossed terms are neglected.

neglecting the region $\varepsilon \to \infty$; in other words, with neglecting $\Sigma'(\mathbf{p}, \varepsilon)$. The linear ε dependence of the self energy derives from the region around ε in the ω integral. This should be cancelled exactly by the term from this region in $\Sigma'(\mathbf{p}, \varepsilon)$. The problem arises because it is falsely believed that this can be neglected in the approximation used. Furthermore it is evident that if the main contributing region of integration to the result is the region explicitly not being dealt with properly then the assumption is invalid.

6.5 Impurity Scattering

To find the resistivity we require the effect that the bosonic propagator has on dressing the impurity vertices. The scattering is from static electronic impurities. We will discuss the scattering integral *only*, as we are interested in the form of this work in the Keldysh formulation. The calculation of the resistivity would follow in the same way[59].

We use the following expression for the resistivity (c.f. the Drude conductivity $\sigma = ne^2\tau/m$)

$$\rho_{res} \sim \tau^{-1} = \Gamma, \tag{6.57}$$

where Γ is the scattering width. This is because only the functional form of the resistivity is derived, which is then fitted to the data using the parameters of the model. However, it should be noted that this approach does not retain any information about the excitations responsible for the conduction. Now the scattering width, Γ, can be written as[69]

$$\Gamma = \int \frac{d\theta}{2\pi}(1 + \cos\theta)|T(\theta)|^2 \approx |T(0)|^2. \tag{6.58}$$

Note that here θ is measured from backscattering. T is the well known T-matrix from scattering theory which can be found from a Dyson's equation using the renormalized scattering amplitudes.

The required equation for the T-matrix is written in terms of the scattering integral A. A is also referred to as the potential matrix and is defined in terms of the impurity potential, $V(\mathbf{r})$, as

$$A(\mathbf{p}_1, \mathbf{p}_2) = \int d\mathbf{r} e^{-i\mathbf{p}_1 \cdot \mathbf{r}} V(\mathbf{r}) e^{i\mathbf{p}_2 \cdot \mathbf{r}}. \tag{6.59}$$

We consider how T, and hence also A, are dressed by the bosonic propagator. The scattering integral is shown in diagram 6.3. For the T-matrix we have

$$T(\theta - \theta', h) = A(\theta - \theta', h) - i\nu_0 \int \frac{d\theta_1}{2\pi} A(\theta - \theta_1, h) T(\theta_1 - \theta', h). \tag{6.60}$$

ν_0 is the Fermi surface density of states as usual. It is clear from equation (6.60) that A is the Born approximation of T. Resolving into angular momentum space we have

$$T_m = \frac{A_m}{1 + iA_m \nu_0} \text{ where} \tag{6.61}$$

$$T(\theta - \theta', h) = \sum_m e^{-im(\theta - \theta')} T_m(h) \text{ and} \tag{6.62}$$

$$A(\theta - \theta', h) = \sum_m e^{-im(\theta - \theta')} A_m(h). \tag{6.63}$$

Now, as $|T_m| \sim \Im T_m$, we finally have

$$\rho_{res} \sim \sum_{m=-\infty}^{\infty} \Im T_m = \Im T_0 + 2 \sum_{m=1}^{\infty} \Im T_m \tag{6.64}$$

as the resistivity. This is the expression that Kim and Millis consider. We will consider the scattering integral.

The lowest order correction to the scattering integral A, dressed by the metamagnetic propagator $P(\mathbf{q}, \omega)$, is figure 6.3(a) and is given by

$$A_1(\varepsilon, \mathbf{p}_1, \mathbf{p}_2) = -\frac{iv_F^2 A_0}{4\pi^2 v p_F} \int d^2q \int d\omega G(P_1 - Q) P(Q) G(P_2 - Q). \tag{6.65}$$

A_0 is the bare scattering integral or potential matrix and is taken to be a constant. We take $P_{1,2} = (\mathbf{p}_{1,2}, \varepsilon)$ and $Q = (\mathbf{q}, \omega)$. Crossed diagrams, such as diagram 6.3(c), are neglected. Higher order noncrossed diagrams can be approximated as

$$A_n = \frac{A_0}{n!}\left(\frac{A_1}{A_0}\right)^n. \tag{6.66}$$

This is due to the result that A_1 depends, approximately, only on the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. Considering we are interested only in small momenta changes and $|\mathbf{p}_1| \approx |\mathbf{p}_2| \approx P_F$ we can write the (backscattering) angle between $\mathbf{p}_1 - \mathbf{q}$ and $\mathbf{p}_2 - \mathbf{q}$ as the same as that between $\mathbf{p}_1$ and $\mathbf{p}_2$. We will also consider the scattering integral to be defined on the Fermi surface. Hence

$$\begin{aligned} A_2(\varepsilon = 0, \mathbf{p}_1, \mathbf{p}_2) &= -\frac{iv_F^2}{4\pi^2 v p_F}\int d^2q d\omega G(P_1 - Q)P(Q) \\ &\quad \underbrace{A_1(\varepsilon - \omega, \mathbf{p}_1 - \mathbf{q}, \mathbf{p}_2 - \mathbf{q})}_{\approx A_1(0,\mathbf{p}_1,\mathbf{p}_2)} G(P_2 - Q) \\ &\approx -\frac{iv_F^2 A_1^2(0, \mathbf{p}_1, \mathbf{p}_2)}{4\pi^2 v p_F A_0} \end{aligned} \tag{6.67}$$

and so on for higher orders. We have

$$A(\alpha, h) = \sum_n A_n(\alpha, h) \tag{6.68}$$

for the total scattering integral, with α the backscattering angle.

6.5.1 Calculation of the Scattering Integral

We need to consider the Keldysh matrix structure of the vertex for the scattering integral. In Keldysh this becomes, to the n^{th} order,

$$\begin{aligned} A_{ij}^n(\mathbf{p}_1, \mathbf{p}_2, \varepsilon) = -\frac{iv_F^2 A_0}{4\pi^2 v p_F}\int d^2q d\omega \gamma_{im}^o G_{mk}(\mathbf{p}_1 - \mathbf{q}, \varepsilon - \omega) A_{kl}^{n-1}(\mathbf{p}_1, \mathbf{p}_2, \varepsilon) \\ G_{ln}(\mathbf{p}_2 - \mathbf{q}, \varepsilon - \omega)\tilde{\gamma}_{nj}^p P_{op}(\mathbf{q}, \omega). \end{aligned} \tag{6.69}$$

Where

$$\gamma^1 = \tilde{\gamma}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \gamma^2 = \tilde{\gamma}^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A^0_{ij} = A_0\delta_{ij}. \tag{6.70}$$

Hence the first order integral to be done is

$$\begin{aligned} A^1_{11} = -\frac{iv_F^2 A_0}{8\pi^2 vp_F}\int & d^2q d\omega [G^R(\mathbf{p}_1 - \mathbf{q}, \varepsilon - \omega)P^K(\mathbf{q}, \omega)G^R(\mathbf{p}_2 - \mathbf{q}, \varepsilon - \omega) + \\ & G^R(\mathbf{p}_1 - \mathbf{q}, \varepsilon - \omega)P^R(\mathbf{q}, \omega)G^K(\mathbf{p}_2 - \mathbf{q}, \varepsilon - \omega) + \\ & G^K(\mathbf{p}_1 - \mathbf{q}, \varepsilon - \omega)P^R(\mathbf{q}, \omega)G^A(\mathbf{p}_2 - \mathbf{q}, \varepsilon - \omega)] \end{aligned} \tag{6.71}$$

with $A^1_{22} = [A^1_{11}]^*$ and $A^1_{21} = 0$ as usual. We expect the Keldysh component to be zero as we are considering an external potential, not an interaction vertex. This is confirmed as A^1_{11} is real and $A^1_{12} \propto [A^1_{11} - A^1_{22}]$. This greatly simplifies equation (6.69) and we can write

$$\mathbf{A}^n(\alpha, h) = \frac{\mathbf{A}_0}{n!}\left(\frac{\mathbf{A}^1(\alpha, h)}{A_0}\right)^n \tag{6.72}$$

as for the zero temperature case discussed in the preceding section.

We define the integral I as

$$A^1_{11}(\alpha, h) = -\frac{iv_F^2 A_0}{4\pi^2 vp_F} I(\alpha, h). \tag{6.73}$$

Thus, the necessary integral is the following:

$$\begin{aligned} I(\alpha, h) = \frac{1}{2}\int d^2q \int d\omega \big[& -2b(\omega)[G^R(P_1 - Q)P^R(Q)G^R(P_2 - Q) \\ & -G^R(P_1 - Q)P^A(Q)G^R(P_2 - Q)] \\ & -2f(\varepsilon - \omega)[G^R(P_1 - Q)P^R(Q)G^R(P_2 - Q) \\ & -G^A(P_1 - Q)P^R(Q)G^A(P_2 - Q)] \\ & +[2G^R(P_1 - Q)P^R(Q)G^R(P_2 - Q) \\ & -G^R(P_1 - Q)P^A(Q)G^R(P_2 - Q) \\ & -G^A(P_1 - Q)P^R(Q)G^A(P_2 - Q)]\big]. \end{aligned} \tag{6.74}$$

$b(\omega)$ is the bosonic distribution function. We will consider small temperatures and break this integral into three stages. Firstly we perform the frequency integral for terms containing the Fermi function $f(\varepsilon-\omega)$. We also define the notation: $\delta = 1, 2$, $\eta_1 = 1$ and $\eta_2 = -1$; and $\xi_\delta = \xi(\mathbf{p}_\delta - \mathbf{q})$. Hence

$$\begin{aligned} I_f &= \int d^2q \int_{\varepsilon}^{\infty} d\omega \frac{ivq}{\xi_1 - \xi_2} \sum_\delta \eta_\delta \frac{1}{\varepsilon - \xi_\delta + ia_q}\left[\frac{1}{-\varepsilon + \omega + \xi_\delta - i\delta} - \frac{1}{-\varepsilon + \omega + \xi_\delta + i\delta}\right] \\ &= \int d^2q \frac{2\pi vq}{\xi_1 - \xi_2} \sum_\delta \eta_\delta \frac{\Theta(-\xi_\delta)}{\varepsilon - \xi_\delta + ia_q}. \end{aligned} \tag{6.75}$$

We deal with the terms including the bosonic distribution function, $b(\omega)$, in a similar way. For this contribution we have

$$\begin{aligned} I_b = \int d^2q \int_{-\infty}^{0} d\omega \frac{ivq}{\xi_1 - \xi_2} \sum_\delta \eta_\delta \Bigg\{ & \frac{1}{\varepsilon - \xi_\delta + ia_q}\left[\frac{1}{\varepsilon_\omega - \xi_\delta + i\delta} + \frac{1}{\omega + ia_q}\right] \\ & - \frac{1}{\varepsilon - \xi_\delta - ia_q}\left[\frac{1}{\varepsilon_\omega - \xi_\delta + i\delta} + \frac{1}{\omega - ia_q}\right]\Bigg\} \\ = \int d^2q \frac{2\pi ivq}{\xi_2 - \xi_1} \sum_\delta \eta_\delta \Bigg\{ & \frac{\varepsilon - \xi_\delta}{(\varepsilon - \xi_\delta)^2 + a_q^2}\left[\frac{1}{\pi}\ln\left[\frac{\varepsilon - \xi_\delta}{a_q}\right] - i\pi\,\Theta(\varepsilon - \xi_\delta)\right] \\ & + \frac{a_q}{2}\frac{1}{(\varepsilon - \xi_\delta)^2 + a_q^2}\Bigg\}. \end{aligned} \tag{6.76}$$

The remaining terms are given by

$$\begin{aligned} I_r = \int d^2q \int_{-\infty}^{\infty} d\omega \frac{ivq}{\xi_1 - \xi_2} \sum_\delta \eta_\delta \Bigg\{ & \frac{1}{\varepsilon - \xi_\delta + ia_q}\left[\frac{1}{\varepsilon - \omega - \xi_\delta + i\delta} + \frac{1}{\omega + ia_q}\right] \\ & + \frac{1}{\varepsilon - \xi_\delta - ia_q}\left[\frac{1}{\varepsilon - \omega - \xi_\delta + i\delta} + \frac{1}{\omega - ia_q}\right] \\ & - \frac{1}{\varepsilon - \xi_\delta + ia_q}\left[\frac{1}{\varepsilon - \omega - \xi_\delta - i\delta} + \frac{1}{-\omega + ia_q}\right]\Bigg\} \\ = \int d^2q \frac{2\pi vq}{\xi_1 - \xi_2} \sum_\delta & \frac{\eta_\delta}{\varepsilon - \xi_\delta + ia_q}. \end{aligned} \tag{6.77}$$

Collecting these three results, we find that

$$\begin{aligned} I = \int d^2q \frac{2\pi vq}{\xi_2 - \xi_1} \sum_\delta \eta_\delta \frac{1}{(\varepsilon - \xi_\delta)^2 + a_q^2} \Bigg[& (\varepsilon - \xi_\delta)\Bigg(\Theta[-\xi_\delta] - 1 + \Theta(\xi_\delta - \varepsilon) \\ & - \frac{i}{\pi}\ln\left[\frac{\varepsilon - \xi_\delta}{a_q}\right]\Bigg) + \frac{ia_q}{2}\operatorname{sgn}(\xi_\delta)\Bigg]. \end{aligned} \tag{6.78}$$

α is the angle between $-\mathbf{p}_1$ and $\mathbf{p}_2$. We will also now set $\varepsilon = 0$ as we restrict ourselves to energies on the Fermi surface.

We note that it is important to include terms of order q^2 in ξ_δ. We describe the result in the absence of these terms in section 6.5.2. We use the idea that $|\mathbf{p}_1| \approx p_F$ and $\mathbf{p}_2 \approx -\mathbf{p}_1$ with α the angle measured from this backscattering position to write

$$\xi_1 = v_F q\cos(\theta) + \frac{q^2}{2m} \qquad \xi_2 = -v_F q\cos(\theta+\alpha) + \frac{q^2}{2m}. \tag{6.79}$$

Hence

$$\begin{aligned} I = \frac{iv}{v_F^2}\int_0^\infty dq\int_0^{2\pi} d\theta\Big[\frac{1}{\cos(\theta)+\cos(\theta+\alpha)} + \frac{1}{\cos(\theta)+\cos(\theta-\alpha)}\Big]\frac{1}{[\cos(\theta)+\frac{q}{2p_F}]^2+b_q^2} \\ \times\Bigg[\big[\cos(\theta)+\frac{q}{2p_F}\big]\ln\Bigg(\frac{b_q^2}{[\cos(\theta)+\frac{q}{2p_F}]^2}\Bigg) - \pi b_q\,\mathrm{sgn}\big[\cos(\theta)+\frac{q}{2p_F}\big]\Bigg]. \end{aligned} \tag{6.80}$$

Now let us use the identity $2\cos(\alpha)\cos(\alpha+\beta) = \cos(\beta) + cos(2\alpha+\beta)$ and shift the θ integral by $\pi/2$. With $xp_F = q$ we find

$$\begin{aligned} I &= \frac{4ivp_F}{v_F^2}\int_0^\infty dx\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin(\theta)}{\sin^2(\theta)-\sin^2(\frac{\alpha}{2})}\frac{1}{(\sin(\theta)+\frac{x}{2})^2+b_x^2} \\ &\quad\times\Bigg[\big(\sin(\theta)+\frac{x}{2}\big)\ln\Bigg[\frac{b_x^2}{(\sin(\theta)+\frac{x}{2})^2}\Bigg] - \pi b_x\big(\sin(\theta)+\frac{x}{2}\big)\Bigg] \\ &\equiv I_n + I_m. \end{aligned} \tag{6.81}$$

I_n is the logarithmically divergent term we are interested in and I_m, as will be shown, can be neglected.

To see that I_m is irrelevant set $\alpha = h = 0$ (where I_n is divergent) and make the usual approximations for small angle and momenta:

$$I_m = -\frac{4\pi ivp_F}{v_F^2}\int_0^\infty dx b_x\int_{-\infty}^\infty dy\frac{1}{y-\frac{x}{2}}\frac{\mathrm{sgn}(y)}{y^2+b_x^2}, \tag{6.82}$$

now rearrange the y integral so the limits are $0 \to \infty$ and substitute $u = y^2$. Then we find

$$I_m = -\frac{32\pi ivp_F}{v_F^2}\int_0^\infty dx\frac{\ln[2x]}{1+4x^2} = 0. \tag{6.83}$$

For I_n we can write

$$\begin{aligned} I_n = \frac{4ivp_F}{v_F^2} \int_0^\infty dx \int_{-\infty}^\infty \frac{dy}{2} \frac{y}{y^2+b^2} \ln\left[\frac{b_x^2}{y^2}\right] \\ \times\left(\frac{1}{y-\frac{x}{2}-\sin(\frac{\alpha}{2})} + \frac{1}{y-\frac{x}{2}+\sin(\frac{\alpha}{2})}\right). \end{aligned} \tag{6.84}$$

We can do the y integral by writing

$$\begin{aligned} \frac{1}{2}\int_{-\infty}^{\infty} dy \frac{1}{y-\gamma}\frac{y}{y^2+b_x^2} \ln\left[\frac{b_x^2}{y^2}\right] &= \int_0^\infty dy \frac{y^2}{y^2-\gamma^2}\frac{1}{y^2+b_x^2} \ln\left[\frac{b_x^2}{y^2}\right] \\ &= -\int_0^\infty du \frac{\sqrt{u}}{u+b_x^2}\frac{1}{u-\gamma^2} \ln\left[\frac{\sqrt{u}}{b_x}\right] \end{aligned} \tag{6.85}$$

and integrating on a contour which goes around a circle "at infinity" and above and below the positive real axis where we place the branch cuts. Then

$$\frac{1}{2}\int_{-\infty}^{\infty} dy \frac{1}{y-\gamma}\frac{y}{y^2+b_x^2} \ln\left[\frac{b_x^2}{y^2}\right] = \int_0^\infty dy \frac{y^2}{y^2-\gamma^2}\frac{1}{y^2+b_x^2} \ln\left[\frac{b_x^2}{y^2}\right] = \frac{\pi^2}{2}\frac{\gamma}{\gamma^2+b_x^2}. \tag{6.86}$$

Combining this result with the necessary γs leaves us with

$$I_n = \frac{2\pi^2 ivp_F}{v_F^2} \int_0^\infty dx \left[\frac{\frac{x}{2}+\sin(\frac{\alpha}{2})}{(\frac{x}{2}+\sin(\frac{\alpha}{2}))^2+b_x^2} + \frac{\frac{x}{2}+\sin(\frac{\alpha}{2})}{(\frac{x}{2}+\sin(\frac{\alpha}{2}))^2+b_x^2}\right] \tag{6.87}$$

to integrate.

In the limits $\alpha \to 0$ and $h \to 0$ this is easy to calculate. If we consider small momenta, where the main contributions to the integral are, we are left with

$$A_{11}^1(\alpha, h) = 2A_0 \ln\left[\frac{1}{\max(h^{\frac{2}{3}}, \alpha^2)}\right]. \tag{6.88}$$

Summing up all the contributions leaves us with

$$\mathbf{A}(\alpha, h) = A_0 \begin{pmatrix} \frac{1}{\max(h^{\frac{4}{3}},\alpha^4)} & 0 \\ 0 & \frac{1}{\max(h^{\frac{4}{3}},\alpha^4)} \end{pmatrix}. \tag{6.89}$$

The power in this expression will in general be dependent on the curvature of the Fermi surface. This generalization can be simply introduced by parameterizing the dispersion as

$\xi_{\mathbf{q}} = v_F q^2/2q_0$[59]. q_0 parameterizes the curvature in the quadratic term in the dispersion (it will not effect the linearized terms). We have concentrated on the case of a circular Fermi surface and would expect different powers from the equivalent calculation of Kim and Millis as we use a slightly different form for the propagator as discussed previously.

6.5.2 Calculation of the Linearized Scattering Integral

If we do not include the quadratic terms in the dispersion relation for the electrons, we find a considerably different result. The logarithmic divergence is replaced by a power law divergence in backscattering angle and applied field. We also destroy the symmetry between these two parameters. First, let us see when the linearization of the dispersion has any validity. From equation (6.80) we can see that, for the linearization of the dispersion to be valid, we require the regions where $\cos(\theta) \gg q/p_F$ to be the most important. However, the important regions of the integrals are where both $\cos(\theta) \approx 0$ and $q/p_F \ll 1$. The difference with the self energy is that both quadratic dispersion terms and contributions from the angular integral are neglected in the same approximation. We show the linearized calculation here to demonstrate that linearization is invalid.

Linearizing the spectrum and taking $\mathbf{p}_1 \approx -\mathbf{p}_2$, with $p_1 \approx p_F$ as before:

$$\xi_1 = v_F q \cos(\theta) \qquad \xi_2 = -v_F q \cos(\theta + \alpha). \tag{6.90}$$

Then we can write equation (6.78) as

$$I = \int d^2q \frac{ivq}{\xi_1 - \xi_2} \sum_\delta \eta_\delta \frac{1}{\xi_\delta^2 + a_q^2} \left[\xi_\delta \ln\left(\frac{a_q^2}{\xi_\delta^2}\right) - \pi a_q \operatorname{sgn}(\xi_\delta) \right] \tag{6.91}$$

or, making the substitution $\theta' = \theta + \alpha - \pi$ in the $\delta = 2$ term,

$$I = \frac{iv}{v_F^2} \int_0^\infty dq \int_0^{2\pi} d\theta \left[\frac{1}{\cos(\theta) + \cos(\theta + \alpha)} + \frac{1}{\cos(\theta) + \cos(\theta - \alpha)} \right] \frac{1}{\cos^2(\theta) + b_q^2} \times \left[\cos(\theta) \ln\left(\frac{b_q^2}{\cos^2(\theta)}\right) - \pi b_q \operatorname{sgn}[\cos(\theta)] \right]. \tag{6.92}$$

Now, for the term containing a logarithm we can substitute $w = \tan(\theta)$. By rearranging the integral we have

$$
\begin{aligned}
I_{ln} &= \frac{2iv}{v_F^2}\int_0^\infty dq \int_0^\pi d\theta \frac{\ln[b^2/\cos^2(\theta)]}{\cos^2(\theta)+b^2}\left\{\frac{1}{1+\cos(\alpha)-\tan(\theta)\sin(\alpha)}\right. \\
&\qquad\qquad\qquad\qquad\qquad \left. +\frac{1}{1+\cos(\alpha)+\tan(\theta)\sin(\alpha)}\right\} \\
&\equiv \frac{2iv}{v_F^2}\int_0^\infty dq[K'(\alpha)+K'(-\alpha)]. \qquad (6.93)
\end{aligned}
$$

Hence, after the substitution,

$$
\begin{aligned}
K'(\alpha) &= -\frac{1}{b_q^2\sin(\alpha)}\int_{-\infty}^\infty dx \frac{\ln[b_q+ib_qw]+\ln[b_q-ib_qw]}{(w-\eta)(w+i\kappa)(w-i\kappa)} \\
&= \frac{2\pi\eta\ln[b_q+\surd(1+b_q^2)]}{b_q^2\sin(\alpha)\kappa(\kappa^2+\eta^2)} - \underbrace{\frac{2\pi\tan^{-1}[\eta]}{b_q^2\sin(\alpha)(\kappa^2+\eta^2)}}_{\to 0 \text{ as } \alpha\to 0 \therefore \text{ it is neglected.}} \quad . \qquad (6.94)
\end{aligned}
$$

Where we have defined

$$\kappa^2 = \frac{1+b^2}{b^2} \qquad \eta = \frac{1+\cos(\alpha)}{\sin(\alpha)}. \qquad (6.95)$$

This integral, equation (6.94), was performed in two parts. The

$$-\frac{1}{b^2\sin(\alpha)}\int_{-\infty}^\infty dx \frac{\ln[1+x^2]}{(x-\eta)(x+i\kappa)(x-i\kappa)} \qquad (6.96)$$

contribution was calculated by defining

$$J(c) = \int_{-\infty}^\infty dx \frac{\ln[c^2+x^2]}{(x-\eta)(x^2+\kappa^2)} \qquad (6.97)$$

and calculating dJ/dc then integrating with respect to c. The remaining terms can be integrated directly. Now we can write

$$I_{ln} = \frac{16\pi iv}{v_F^2}\int dq \frac{\ln[b_q+\surd(1+b_q^2)]}{4b_q^2+\alpha^2}\frac{b_q}{\surd(1+b_q^2)} \qquad (6.98)$$

for this contribution.

For small momenta and applied field ($b \ll 1$) we can approximate I_{ln} as

$$I_{ln} \approx \frac{16\pi i v}{v_F^2} \int dq \frac{b_q}{4b_q^2 + \alpha^2} \ln[1 + b_q]. \tag{6.99}$$

Hence

$$I_{ln}(\alpha = h = 0) \approx \frac{4\pi i \sqrt{v}}{v_F^{\frac{3}{2}} \xi_0} \underbrace{\int_0^\infty \frac{dx}{x^2} \ln[1 + x^2]}_{=\pi} \tag{6.100}$$

and we can safely neglect this contribution as the remaining terms, in equation (6.92), contain divergences.

Keeping the α dependence explicit, in the remaining terms we have,

$$I = -\frac{\pi v}{v_F^2} \int dq d\theta \frac{1}{\cos(\theta) + \cos(\theta + \alpha)} \left[\frac{i b_q \operatorname{sgn}[\cos(\theta)]}{\cos^2(\theta) + b_q^2} - \frac{i b_q \operatorname{sgn}[-\cos(\theta + \alpha)]}{\cos^2(\theta + \alpha) + b_q^2} \right]. \tag{6.101}$$

If we shift the appropriate parts of the θ integral by π and α we can write this as

$$\begin{aligned} I = \frac{2\pi v}{v_F^2} \int dq \Bigg[& \int_{\pi/2}^{3\pi/2} d\theta \frac{i b_q}{\cos(\theta) + \cos(\theta + \alpha)} \frac{1}{\cos^2(\theta) + b_q^2} + \\ & \int_{\pi/2}^{3\pi/2} d\theta \frac{i b_q}{\cos(\theta) + \cos(\theta - \alpha)} \frac{1}{\cos^2(\theta) + b_q^2} \Bigg]. \end{aligned} \tag{6.102}$$

We split this into integrals such that

$$I \equiv \frac{2\pi v}{v_F^2} \int dq [i b_q (K(\alpha) + K(-\alpha))]. \tag{6.103}$$

Now we can calculate $K(\alpha)$:

$$\begin{aligned} K(\alpha) = \frac{1 + \cos(\alpha)}{2b_q^2[1 + \cos(\alpha)] + \sin^2(\alpha)} \Bigg\{ & \frac{1}{\sqrt{1 + b_q^2}} \ln \left[\frac{1 + \sqrt{1 + b_q^2}}{1 - \sqrt{1 + b_q^2}} \right] \\ & - \sqrt{2 + 2\cos(\alpha)} \ln \left[\frac{1 + \sqrt{2 + 2\cos(\alpha)}}{1 - \sqrt{2 + 2\cos(\alpha)}} \right] \Bigg\} \end{aligned} \tag{6.104}$$

which for small angles gives

$$K(\alpha) = \frac{2}{4b_q^2 + \alpha^2} \left[\frac{1}{\sqrt{1 + b_q^2}} \ln \left(\frac{1 + \sqrt{1 + b_q^2}}{1 - \sqrt{1 + b_q^2}} \right) - \ln \left(\frac{16}{\alpha^2} \right) \right]. \tag{6.105}$$

This integral was performed by the series of substitutions: $w = \tan(\theta)$; $u = w^{-2}$; and $1 + u = y^2$. Hence we find

$$I = \frac{2\pi v}{v_F^2} \int dq \frac{4ib_q}{4b_q^2 + \alpha^2} \left[\frac{1}{\sqrt{1+b_q^2}} \ln \left(\frac{1+\sqrt{1+b_q^2}}{1-\sqrt{1+b_q^2}} \right) - \ln \left(\frac{16}{\alpha^2} \right) \right] \tag{6.106}$$

for this integral.

The most important parts of this integral come from small momenta so we may approximate $b \ll 1$ in the integral, thus

$$I \approx \frac{8\pi i v}{v_F^2} \int_0^\infty dq \frac{b}{4b^2 + \alpha^2} \ln \left[\frac{\alpha^2}{4b^2} \right]. \tag{6.107}$$

Setting $h \to 0$ and rescaling we find equation (6.107) to be

$$I(\alpha > h \to 0) = \frac{4\pi i v \sqrt{v_F}}{\xi_0 v_F^2 \sqrt{(2v)}} \frac{1}{\sqrt{\alpha}} \int_0^\infty dx \frac{x^2}{1+x^4} \ln \left[\frac{1}{x^4} \right] \propto \frac{i}{v_F \xi_0} \frac{1}{\sqrt{\alpha}}. \tag{6.108}$$

The integral is just a number and we have found a power law divergence. Setting $\alpha \to 0$ in equation (6.107) (where we can) we find

$$I(h > \alpha \to 0) = \frac{\pi^2 i}{v_F \xi_0} \frac{1}{h^{\frac{1}{3}}} \ln \left[\frac{\alpha^2}{h^{\frac{2}{3}}} \right], \tag{6.109}$$

as $h \to 0$. Hence for the first order linearized scattering integral we have

$$A_{11}^1(\alpha > h = 0) = \frac{\pi A_0 \sqrt{v_F}}{4 p_F \xi_0 \sqrt{v}} \frac{1}{\sqrt{\alpha}} \tag{6.110}$$

$$A_{11}^1(h > \alpha \to 0) = -\frac{A_0 v}{2 p_F \xi_0 v_F} \frac{1}{h^{\frac{1}{3}}} \ln \left[\frac{h^{\frac{2}{3}}}{\alpha^2} \right]. \tag{6.111}$$

The symmetry between backscattering angle and applied field has been destroyed in this regime. We also have a considerably worse divergence. This demonstrates the importance, in contrast with the self energy term, of including quadratic dispersion elements.

6.5.3 Comparison with Kim and Millis' Scattering Integral

Kim and Millis find the result to be

$$A_1(\alpha, h) = -A_0 \, \mathrm{I}\left(\frac{2\pi q_0}{2\pi p_F}\right) \ln[\max(h^{\frac{2}{3}}, \alpha^2)]. \tag{6.112}$$

Where I is an integral depending on the shape and size of the Fermi surface given by

$$\mathrm{I}(b) = \frac{2b}{\pi}\int_0^\infty dy \frac{\frac{2by}{\sqrt{3}}}{(1+y^3)\left[\left(\frac{2b}{\sqrt{3}}\right)^2 + y^4\right]}. \tag{6.113}$$

q_0 parameterizes the curvature of the Fermi surface. Summing all orders leaves the power law

$$A(\alpha, h) = A_0 \frac{1}{[\max(h^{\frac{2}{3}}, \alpha)]^I}. \tag{6.114}$$

With, for a circular Fermi surface, $I \approx 0.23$[59]. We differ from this result only in terms of the power law due to using different forms for the bosonic propagator. We have also neglected the considerations of parameterizing the curvature of the Fermi surface.

Using equations (6.61), (6.64) and (6.114) the residual (in other words zero temperature) resistivity is calculated. This is measured and calculated as a function of the applied field. Their results, shown in figure 6.4, give the peak in the resistivity. From the height of the peaks they claim they require a value for I of ≈ 0.75. However, this is not a fit of the shape of the plots. Note that these plots are of the residual resistivity as we are at zero temperature. Experimental data[52] is given in figure 6.5.

6.6 Summary

Summarizing, in this chapter we have addressed a phenomenological model for metamagnetic quantum criticality. We presented what assumptions were necessary to derive this model. Following this we looked at the self energy and corrections to the scattering amplitude in such a model, which was presented in the Keldysh technique. We found some differences between our approach and that of Kim and Millis in the original paper.

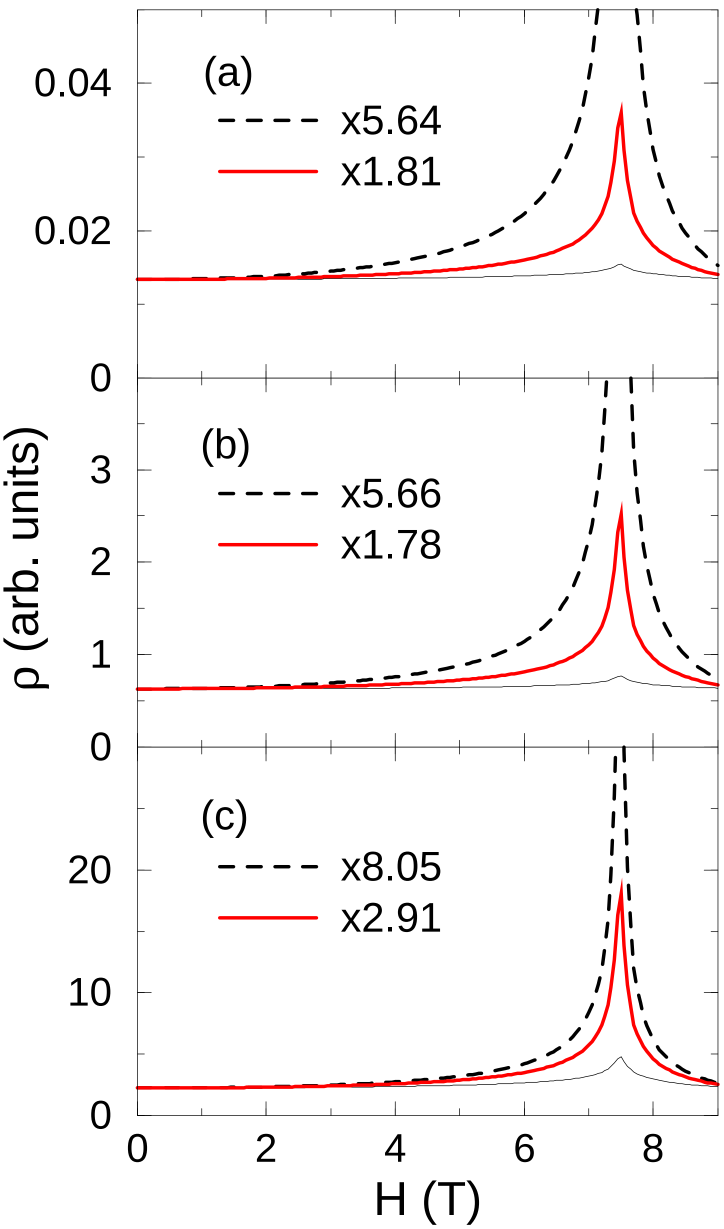


Figure 6.4: From [59], the residual resistivity calculated by Kim and Millis. The different lines refer to different values of I. $I = 1$ are the dashed lines, $I = 0.75$ are the solid lines and $I = 0.5$ are the light solid lines. The three panels are for different initial scattering amplitudes: (a) is $A_0\nu_0 = 0.5$, (b)is $A_0\nu_0 = 1.0$ and (c) is $A_0\nu_0 = 5.0$.

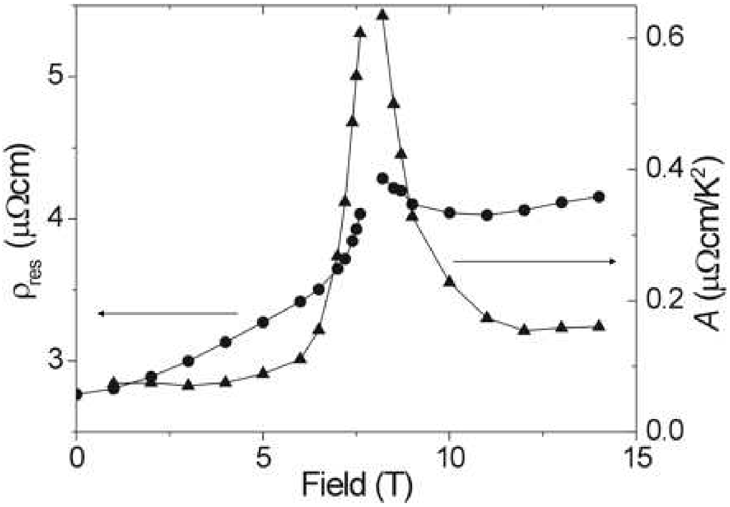


Figure 6.5: From [52], the residual resistivity (triangles). Also plotted is the coefficient A in the formula $\rho = \rho_{res} + AT^2$.

Chapter 7

CONCLUSION

7.1 Discussion of the Coulomb Blockade

In the first half of this thesis we have found expressions for the tunnelling density of states and the conductance of a quantum dot using the Keldysh technique. We began by introducing the concepts involved with mesoscopic physics and the interplay of electron-electron interactions and interference due to disorder. Specifically, we consider the zero-bias anomaly in quantum dots. We show how this can be perturbatively explained by considering the lowest order corrections due to the interaction and disorder averaging. This is generalized to the nonequilibrium Keldysh diagrammatic technique. For temperatures below the charging energy, $E_c \sim e^2/C$, this perturbative technique becomes invalid and we need to consider the zero-momentum interaction mode nonperturbatively.

This effect, caused by the dominant zero momentum mode, is the Coulomb blockade. The first part of this thesis is dominated by our explanation of this phenomenon. Our approach is to consider a functional integral representation for the single particle Green's function. The interaction can be dealt with by introducing a Hubbard-Stratonovich transformation which allows us to deal exactly with most parts of the bosonic field categorizing the interaction. The exact manipulation can be performed either with a gauge transformation or by direct integration. However, it is not possible to deal exactly with the zero-mode of the bosonic field. By zero-mode we here refer to, in the Matsubara technique, the zero frequency component. It is not possible to remove this component by the gauge transformation. In the Keldysh technique the analogous part of the bosonic field is the integral over the interaction contour of the field.

Previous calculations[24, 35] apply saddle point approximations to the bosonic field's zero-mode. We demonstrate how to do this in our calculation to explain the differences in the approaches. We also consider an alternative derivation which highlights the interpretation of the result of the Green's function. This is expressed as a sum over N-particle canonical ensembles.

The saddle point solution we use has an infinite number of solutions. We note that it is important to include the whole set of saddle point solutions for the Green's function to correctly describe the necessary phenomena. This has not been correctly taken into account by Kamenev and Gefen in their formulation[24]. A different attempt by Efetov and Tschersich[35] correctly describes the behaviour at the Coulomb peaks but not in the valleys. This includes the infinite number of saddle point solutions in terms of winding numbers of the gauge field introduced.

From the single particle Green's functions we have the tunnelling density of states. The use of the full saddle point solution gives us a previously unknown effect of the Coulomb blockade. At the peak sites we discover that the tunnelling density of states is suppressed to half of the free particle density of states. Previously it had been thought that it is flat at the peaks (and of the free particle value). In the valleys we find the tunnelling density of states is fully suppressed, as is expected.

In the calculation of Kamenev and Gefen, the Coulomb blockade is not properly accounted for. The suppression of the density of states they find is the remnant of the high temperature, $T \gg E_c$, zero-bias anomaly. In this limit we recover their result exactly. This can also be seen by checking the result for the average number of particles on the dot. Their saddle point solution describes a linear change of $\langle N \rangle$ with changing gate voltage.

We can also consider the behaviour of the average number of particles on the dot, $\langle N \rangle$, with changing gate voltage. We find that, as the gate voltage is altered, the average particle number jumps suddenly by one. This occurs at the values $E_c(n + 1/2)$, as is expected. ($\{n\}$ are the integers.) This is plotted in figure 1.3.

Furthermore, to correctly gain the expression for conductance (compare with [8]) we need to correctly describe the tunnelling density of states at the degeneracy point (the half gap). We consider the model of two quasi-one dimensional wires weakly coupled to the quantum dot. We require weak coupling to ensure that the particle number on the dot remains a good quantum number. The one dimensional wires are treated as reservoirs in equilibrium, and we are interested in the steady state current.

To summarize, we consider the zero-bias anomaly and the role of the zero-momentum mode of the interaction. We then calculate the Green's function for an isolated quantum dot. From this we find novel behaviour of the tunnelling density of states and derive the differential conductance across an almost closed dot.

7.2 Discussion of the Metamagnetic Quantum Critical Point

In the second part of this thesis we have addressed the phenomenon of metamagnetic quantum criticality. We briefly introduce the idea of a quantum critical point and quantum critical endpoints. These were first introduced by Hertz[46] in 1976. The models we are interested in were brought back to prominence by Millis[55] in 1993.

The introduction of a quantum critical point referred to the specific case of metamagnetism. This is a magnetic phase transition which consists of a discontinuity in the magnetization at a specific applied field strength. The system starts as a paramagnet at low applied fields. For the case we are interested in this is around 7 Tesla. The metamagnetic phase transition is first order and has an end point at a finite temperature.

Metamagnetism is a phenomenon observed in the crystal $Sr_3Ru_2O_7$, and we present the experimental evidence for this case. It can also be shown that the critical end point of the metamagnetic phase transition can be tuned to zero temperature. This is performed by changing the angle of the applied magnetic field with the crystal. When the field approaches being perpendicular to the planes, the critical end point goes toward absolute zero and we have a quantum critical end point. It is the behaviour around this quantum critical end point that we are interested in.

The model we are considering consists of electrons with a spin density interaction which is thought to describe the metamagnetic quantum critical point. We demonstrate how this model is derived and show the resulting action and the bosonic propagators which describe its excitations.

This metamagnetic model is modified by Kim and Millis[59] by coupling these excitations with free electrons in the system. Though strictly speaking this model double counts the number of electrons in the system we demonstrate what assumptions are necessary to derive it by assuming that there exist two classes of electrons in the material. We require that these electrons interact via a spin density interaction. Also one set of electrons must be otherwise free and the other set interact with a spin density interaction. The second set of electrons gives rise to the usual metamagnetic bosonic action. The first set, which must exist in the a-b planes of the material, should be the electrons which are involved in the transport measurements. Most importantly the two classes of electrons, A and B, must be uncorrelated such that $\langle\bar{\psi}^A\psi^B\rangle = 0$ and $\langle\bar{\psi}^B\psi^A\rangle = 0$.

We re-calculate the results for this model using the Keldysh diagrammatic technique. However we believe there to be several mistakes in their calculation. One trivial error leads to incorrect numerical coefficients for some results. This is the use of the incorrect zero temperature form for the bosonic propagator. They incorrectly move from the finite temperature Matsubara form (in which their model is derived) to the zero temperature form.

However, a more important problem is a mistake in their derivation of the self energy correction to an electron interacting with the bosonic field. From the self energy we find the lifetime of the quasiparticle for the system. They use an approximation which mistreats one of the most important parts of the frequency integral. This gives the correct lifetime in one limit (applied field $h = 0$ and frequency $\varepsilon \neq 0$) but not in the other (frequency $\varepsilon \to 0$ and applied field $h \neq 0$). In the second limit we calculate that the lifetime is $\sim -(\varepsilon/h^{\frac{2}{3}})^2 \ln(\varepsilon)$. This is a modified Fermi liquid result which simply gives a quadratic dependence on frequency. However, they find that the lifetime is $\sim \varepsilon/h^{\frac{1}{3}}$.

Further to this, we consider the effect of the metamagnetic propagator on the scattering integral, and it is found to give a power law divergence. This divergence is in either the angle from backscattering between incoming and outgoing electrons, or in the applied field measured from the metamagnetic phase transition. We agree with the result of Kim and Millis which we calculate using the Keldysh formalism, and we note the importance of not linearizing the electron dispersion in this scattering integral.

Appendix A

CONDUCTIVITY

The quantum mechanical expression for current[18] is given by

$$\begin{aligned} \hat{\mathbf{j}}(r) &= \hat{\mathbf{j}}(r, r')|_{r=r'} \\ &= \frac{ie}{2m}[(\nabla_r - \nabla_{r'})|r\rangle\langle r'|]_{r=r'}. \end{aligned} \tag{A.1}$$

Now, using the matrix elements for $\mathbf{j}_{nm}$ we obtain

$$\begin{aligned} \mathbf{j}_{nm}(r) &= \langle n|\hat{\mathbf{j}}(r)|m\rangle \\ &= \frac{ie}{2m}\{[\nabla\psi_n^*(r)]\psi_m(r) - \psi_n^*(r)[\nabla\psi_m(r)]\}. \end{aligned} \tag{A.2}$$

We now wish to use linear response theory to express this in terms of free particle Green's functions and the vector potential.

The Hamiltonian for an electron in a vector potential A is substituted into the formula for the Green's function and S is then expanded to linear order in A. (It is also possible to find linear response from the Kubo formula[56].)

For a metal in an electric field E the current is $\mathbf{j} = \sigma\mathbf{E}$. Using basic electrodynamic formulae

$$\mathbf{j}_\omega = i\omega\sigma A_\omega, \tag{A.3}$$

$$\mathbf{j}_\omega = -Q(\mathbf{p}, \omega)A_\omega \text{ and} \tag{A.4}$$

$$Q(\mathbf{p}, \omega) = -i\omega\sigma, \tag{A.5}$$

for the kernel Q, the conductivity is found from the current density in momentum space. As the Green's function for a particle in an electric field is:

$$
\begin{aligned}
G(x, x') &= -i\langle \mathrm{T}\, \hat{\psi}_H(x)\hat{\psi}^\dagger_H(x')\rangle; && \text{(A.6)} \\
&= -i\frac{\langle \mathrm{T}\, S(\infty)\hat{\psi}(x)\hat{\psi}^\dagger(x')\rangle}{\langle S(\infty)\rangle}, && \text{(A.7)} \\
\text{with } S(\infty) &= \mathrm{T}\exp\big(-i\int_{-\infty}^{\infty} H_i(t'')dt''\big), && \text{(A.8)} \\
H_i &= \frac{1}{2}\int d\mathbf{r}\mathbf{j}.\mathbf{A} && \text{(A.9)} \\
\text{and } \mathbf{j}.\mathbf{A} &= -\frac{ie}{2m}\mathbf{A}.(\nabla_{\mathbf{r}} - \nabla_{\mathbf{r}'})_{\mathbf{r}'\to\mathbf{r}}[\psi(x)\psi^\dagger(x')]_{t'\to t} + O(\mathbf{A}^2). && \text{(A.10)}
\end{aligned}
$$

Hence we can write the current of the system, $\mathbf{J}$, as

$$
\mathbf{J}(x) = -\frac{ie^2}{2m^2}(\nabla_{\mathbf{r}'} - \nabla_{\mathbf{r}})_{\mathbf{r}'\to\mathbf{r}}[G(x, x')]_{t'\to t}. \tag{A.11}
$$

Expanding S and applying wick's theorem, to linear order in A, we find

$$
\begin{aligned}
\mathbf{J}(x) = -\frac{ie^2}{2m^2}(\nabla_{r'} - \nabla_{r})_{r'\to r}\int d^4y\mathbf{A}(y)(\nabla_y - \nabla_{y'})_{y'\to y}G(x, y')G(y, x') \\
-\frac{Ne^2}{m}\mathbf{A}(x).
\end{aligned} \tag{A.12}
$$

Where $y = (\mathbf{y}, t')$, $y' = (\mathbf{y}', t')$ $x = (\mathbf{r}, t)$ and $x' = (\mathbf{r}', t)$. After Fourier transforming, the conductivity can be extracted as

$$
\sigma(\mathbf{k}, \omega) = \frac{iNe^2}{\omega m} + \frac{2e^2}{\omega m^2}\int \frac{d^d\mathbf{p}'}{(2\pi)^d}\frac{d\epsilon}{2\pi}\mathbf{p}\mathbf{p}'\langle G(\mathbf{p}_+, \mathbf{p}'_+; \epsilon + \frac{1}{2}\omega)G(\mathbf{p}'_-, \mathbf{p}_-; \epsilon - \frac{1}{2}\omega)\rangle_i. \tag{A.13}
$$

$\mathbf{p}_\pm = \mathbf{p}\pm\frac{\mathbf{k}}{2}$. We have now included impurity averaging explicitly and the Green's functions include an impurity potential. As the impurity averaging has not been performed yet the system is not spatially invariant and so momentum is not conserved. (This is why the Green's functions have two momenta coordinates but only one frequency coordinate.)

Labelling

$$
\Xi(P_+, P_-) = \frac{1}{(2\pi)^3}\int G(\mathbf{p}_+, \mathbf{p}'_+; \omega_+)G(\mathbf{p}'_-, \mathbf{p}_-; \omega_-)\mathbf{p}'d\mathbf{p}' \tag{A.14}
$$

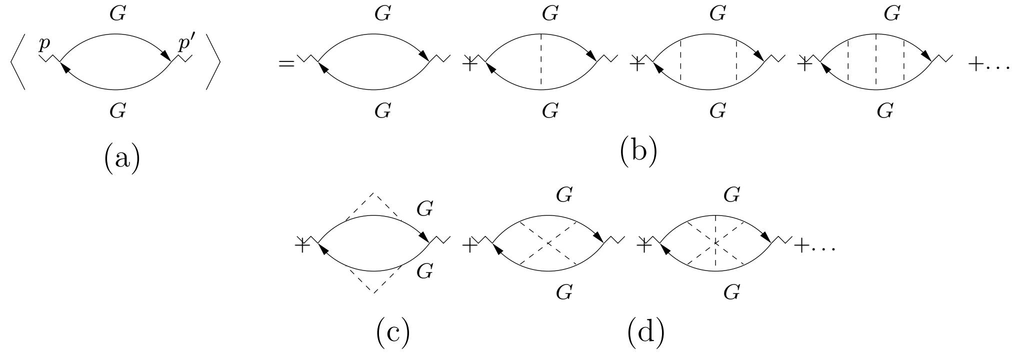


Figure A.1: Conductivity contributions. The cooperon sum is (d). (c) shows the contribution for the Drude conductivity σ_0.

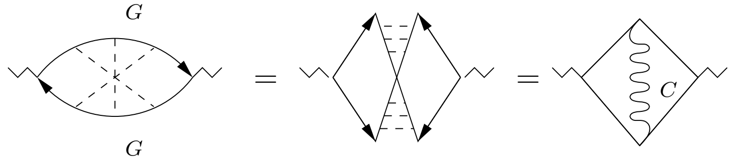


Figure A.2: Conductivity contribution from a cooperon, the sum in figure A.1(d). Sums over all such crossed diagrams is implied.

we can perform impurity averaging on this object and use Dyson's equation to sum the low order contributions. This leads to the equation[37]:

$$\Xi(P_+, P_-) = G(P_+)G(P_-)\left[\mathbf{p} + \frac{1}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \Xi(P'_+, P'_-)d\mathbf{p}'\right]. \tag{A.15}$$

With u the impurity potential. This integral can be calculated in the normal skin effect, $\nu|\mathbf{k}| \ll 1/\tau$[37, 56].

Having performed impurity averaging for the conductivity diagram, see figure A.1, via the above method we have for the lowest order correction

$$\sigma = \frac{2ie^2}{m^2}\frac{1}{\tau}\frac{1}{\omega + \frac{i}{\tau_{tr}}}\int \frac{d^dp}{(2\pi)^d}\frac{d\epsilon}{2\pi}\mathbf{p}.\mathbf{p}' G(p'_+)G(p'_-). \tag{A.16}$$

With τ_{tr} the transport lifetime: a phenomenological constant associated with the above calculation of Ξ. This will give the Drude conductivity.

It has been shown that the maximally crossed diagrams, figure A.1(d), give a singular contribution which we are interested in here. The maximally crossed diagrams can be rewritten as a cooperon contribution. With $\omega \approx 0$, static response, the cooperon will simply give a contribution like $1/\mathbf{q}^2$, from section 2.2.2. See figure A.2, and so this is given by

$$\sigma = \frac{2e^2}{\pi L^d} \sum_{\mathbf{q}} \frac{1}{\mathbf{q}^2}. \tag{A.17}$$

The green's functions in the diagram can be trivially integrated out to lowest order.

Appendix B

EQUIVALENT DIAGRAMS FOR THE DIFFUSONS AND COOPERONS

In this appendix we present some equivalent diagrams which are used for the Diffuson and Cooperon modes, labeled D and C respectively. See figure B.1. Also given is an identical form for a scattering event that has been used. Note that the points for a scattering event are concurrent in position but not necessarily in time. (Averaging over disorder gives delta functions in space as the correlated scattering events are off the same immobile impurity but they can happen at different times.)

The ladder diagram for the diffuson is then figure B.2. Which is

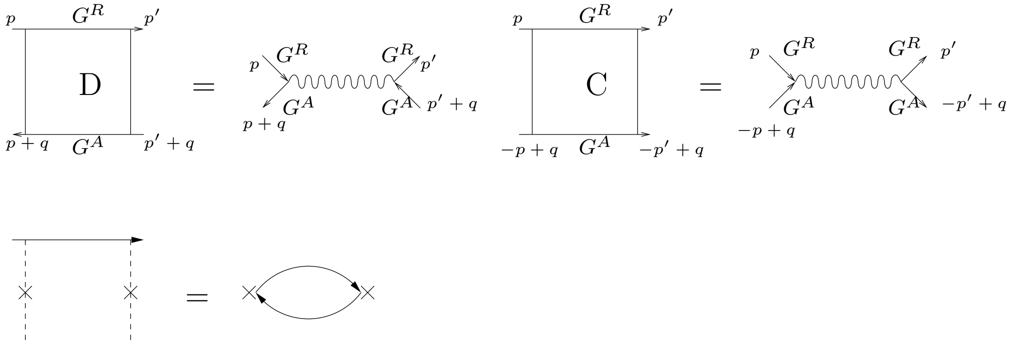


Figure B.1: Further diffuson and cooperon diagrams.

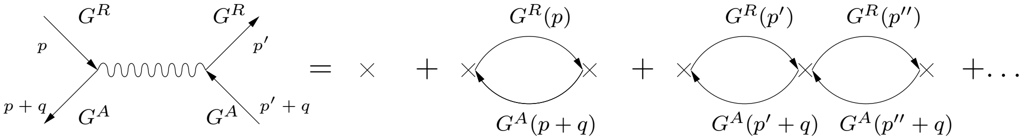


Figure B.2: Diffuson ladder diagram.

$$\begin{aligned} D(q,\omega) &= \frac{1}{2\pi\nu_d\tau} + \frac{1}{2\pi\nu_d\tau}K_0\frac{1}{2\pi\nu_d\tau} + \frac{1}{2\pi\nu_d\tau}K_0\frac{1}{2\pi\nu_d\tau}K_0\frac{1}{2\pi\nu_d\tau} + \cdots \\ &= \frac{1}{2\pi\nu_d\tau}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{K_0}{2\pi\nu_d\tau}\right)^n \\ &= \frac{1}{2\pi\nu_d\tau^2(Dq^2 - i\omega)}. \end{aligned} \tag{B.1}$$

Appendix C

DERIVATION OF THE POLARIZATION OPERATOR

The polarization operator is defined as

$$\Pi^{kk'}(x,x') = 2i\tilde{\gamma}^{k}_{j'i'}\langle G_{i'i}(x,x')G_{jj'}(x',x)\rangle\gamma^{k'}_{ij}. \tag{C.1}$$

Now to check that this has the correct Keldysh form we wish to check that $\Pi^{21}=0$. This is trivial to perform by Fourier transforming with respect to time and noting that

$$\begin{aligned}\Pi^{21}(x,x') = i\langle\int d\varepsilon[&G_{11}(\mathbf{r},\mathbf{r}';\varepsilon)G_{11}(\mathbf{r}',\mathbf{r};\varepsilon+\omega)\\ &+G_{22}(\mathbf{r},\mathbf{r}';\varepsilon)G_{22}(\mathbf{r}',\mathbf{r};\varepsilon+\omega)]\rangle = 0,\end{aligned} \tag{C.2}$$

by the analytic properties of the advanced and retarded Green's functions.

Now, the retarded component of the polarization operator is

$$\Pi^{11}(x,x') = i\langle G^R(x,x')G^K(x',x) + G^K(x,x')G^A(x',x)\rangle_i. \tag{C.3}$$

The advanced component will simply be the complex conjugate of this. After Fourier transforming with respect to $(t-t')$, as $G(x,x')=G(\mathbf{r},\mathbf{r}';t-t')$, this becomes

$$\begin{aligned}\int\frac{d\varepsilon}{2\pi}\big(&G^R(\varepsilon)G^A(\varepsilon+\omega)[h(\varepsilon)-h(\varepsilon+\omega)] + h(\varepsilon+\omega)G^R(\varepsilon)G^R(\varepsilon+\omega)\\ &-h(\varepsilon)G^A(\varepsilon)G^A(\varepsilon+\omega)\big).\end{aligned} \tag{C.4}$$

For clarity we have dropped the position dependence (which is easy to put back provided we respect the order of the Green's functions) and the impurity averaging. By Taylor expanding we can use, for small ω, $h(\varepsilon)-h(\varepsilon+\omega)\approx-\omega\partial_\varepsilon h(\varepsilon)$. As h is a step like function its derivative will simply set the boundaries on the integral over ε. Then, integrating the first term of equation (C.4) leads to

$$-\frac{\omega}{2\pi}[G^R(\varepsilon)G^A(\varepsilon)+\omega)h(\varepsilon)|_{-\infty}^{\infty}]+\frac{\omega}{2\pi}\int d\varepsilon h(\varepsilon)\frac{\partial}{\partial\varepsilon}[G^R(\varepsilon)G^A(\varepsilon+\omega)]. \tag{C.5}$$

The impurity averaging and momentum integration for the first of these two terms has already been calculated for the diffuson. The second and third terms of equation (C.4) are calculated by using the step like properties of the distribution functions and the property $\partial_\varepsilon G^{R/A}(\varepsilon)=-[G^{R/A}(\varepsilon)]^2$. ω is set to zero in these terms as we are only interested in the behaviour for small frequencies.

Thus we now have

$$\begin{aligned}\Pi^{11}(\mathbf{q},\omega)=&\ \nu_d+\frac{i\omega\nu_d}{D\mathbf{q}^2-i\omega}\\&+\frac{i\omega}{2\pi}\int d^d(r-r')e^{i\mathbf{q}.(\mathbf{r}-\mathbf{r}')}\langle\int d\varepsilon h(\varepsilon)\partial_\varepsilon[G^R(\mathbf{r},\mathbf{r}';\varepsilon)G^A(\mathbf{r}',\mathbf{r};\varepsilon+\omega)]\rangle.\end{aligned}\tag{C.6}$$

The last term has an integrand of order ε^{-3} as $\varepsilon\to\pm\infty$ and so is neglected in comparison with the first two terms. Hence

$$\Pi^{11}(\mathbf{q},\omega)=\frac{D\mathbf{q}^2\nu_d}{D\mathbf{q}^2-i\omega} \tag{C.7}$$

is the retarded component of the Polarization operator.

For the Keldysh component we have

$$\Pi^{12}(\mathbf{r}-\mathbf{r}',\omega)=i\int\frac{d\varepsilon}{2\pi}\langle G^R(\varepsilon)G^A(\varepsilon+\omega)+G^A(\varepsilon)G^R(\varepsilon+\omega)+G^K(\varepsilon)G^K(\varepsilon+\omega)\rangle \tag{C.8}$$

We can rearrange this into

$$\Pi^{12}(\mathbf{r}-\mathbf{r}',\omega)=i\int\frac{d\varepsilon}{2\pi}[h(\varepsilon)h(\varepsilon+\omega)-1]\langle(G^R-G^A)_\varepsilon(G^R-G^A)_{\varepsilon+\omega}\rangle. \tag{C.9}$$

Performing this integral and the impurity averaging, which is the previously calculated $\langle G^R G^A \rangle_i$, we have

$$\Pi^{12}(\mathbf{q},\omega) = \underbrace{\frac{1}{\omega}\int d\varepsilon[1 - h(\varepsilon)h(\varepsilon+\omega)]}_{\equiv I(\omega)}[\Pi^R(\mathbf{q},\omega) - \Pi^A(\mathbf{q},\omega)] \tag{C.10}$$

for the Keldysh component of the polarization operator.

Appendix D

KELDYSH VERTEX CORRECTIONS

The Keldysh vertices can be rewritten to include the scattering from impurities across Green's functions either side of the emitted or absorbed Coulomb propagator. The absorption vertex Γ^k_{ij} is

$$\mathbf{\Gamma}^1(\mathbf{q},\omega;\varepsilon) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \frac{h(\varepsilon+\omega)-h(\varepsilon)}{(Dq^2-i\omega)\tau} \\ 0 & 1 \end{pmatrix} \text{ and} \tag{D.1}$$

$$\mathbf{\Gamma}^2(\mathbf{q},\omega;\varepsilon) = \frac{1}{\sqrt{2}(Dq^2+i\omega)\tau}\begin{pmatrix} -h(\varepsilon+\omega) & \frac{2(h(\varepsilon+\omega)h(\varepsilon)-1)}{(Dq^2-i\omega)\tau} \\ 1 & h(\varepsilon) \end{pmatrix}. \tag{D.2}$$

The emission vertex $\widetilde{\Gamma}^k_{ij}$ is

$$\widetilde{\mathbf{\Gamma}}^1(\mathbf{q},\omega;\varepsilon) = \frac{1}{\sqrt{2}(Dq^2-i\omega)\tau}\begin{pmatrix} -h(\varepsilon-\omega) & \frac{2(h(\varepsilon-\omega)h(\varepsilon)-1)}{(Dq^2+i\omega)\tau} \\ 1 & h(\varepsilon) \end{pmatrix} \text{ and} \tag{D.3}$$

$$\widetilde{\mathbf{\Gamma}}^2(\mathbf{q},\omega;\varepsilon) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \frac{h(\varepsilon-\omega)-h(\varepsilon)}{(Dq^2+i\omega)\tau} \\ 0 & 1 \end{pmatrix}. \tag{D.4}$$

However in practice it is often easier not to use these averaged vertices but to construct the diagrams from the basic elements and unavereged vertices which have a simpler structure. Impurity averaging can then be performed which can not alter the Keldysh matrix structure.

The above are calculated in a similar way to zero temperature vertex corrections, simply including the matrix structure and distribution functions of the Keldysh method.

Appendix E

THE HUBBARD-STRATONOVICH TRANSFORMATION

The Hubbard-Stratonovich transformation can be used to transform quartic terms in the functional integral action and replace them with quadratic terms, at the expense of introducing a new field to integrate over. We also note some subtle issues associated with the requirement that our Hamiltonian is normal ordered before we derive the functional integral.

The Hubbard-Stratonovich transformation for a general normal ordered interaction, $\frac{1}{2}\operatorname{Tr}[\bar{\psi}\bar{\psi}V\psi\psi]$, is

$$\begin{aligned}\int \frac{D\phi}{\mathcal{N}} e^{-\frac{i}{2}\operatorname{Tr}\phi V^{-1}\phi + i\operatorname{Tr}\bar{\psi}\phi\psi} &= \int \frac{D\phi}{\mathcal{N}} e^{-\frac{i}{2}\operatorname{Tr}[\phi - \bar{\psi}\psi V]V^{-1}[\phi - V\bar{\psi}\psi]} e^{-\frac{i}{2}\operatorname{Tr}\bar{\psi}\bar{\psi}V\psi\psi} \\ &= e^{-\frac{i}{2}\operatorname{Tr}\bar{\psi}\bar{\psi}V\psi\psi}. \qquad \text{(E.1)}\end{aligned}$$

(Matrix multiplication is implied over all arguments of the necessary fields.) Note that if ψ is a fermionic field then it is required that ϕ is bosonic and it must obey periodic boundary conditions on the appropriate time contour.

In our method for re-exponentiating after performing the functional integrals, see equation (2.71) we need to be careful. If we do not include terms up to the correct order we introduce additional unwanted terms, or rather fail to cancel unwanted terms which should not be present. Consider the toy Hamiltonian $\frac{1}{2}E\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c}$. The partition function is clearly, due to the fermionic properties of $\hat{c}$,

$$Z = \operatorname{Tr} e^{\frac{1}{2}E\hat{c}^\dagger\hat{c}^\dagger\hat{c}\hat{c}} = \operatorname{Tr} e^0 = 2. \qquad \text{(E.2)}$$

In terms of the functional integral this becomes

$$Z = \int D\psi D\bar{\psi} e^{i\int_c dt(\bar{\psi}i\partial_t\psi - \frac{1}{2}E\bar{\psi}^2\psi^2)} \tag{E.3}$$

$$= \int \frac{D\phi}{\mathcal{N}_\phi} e^{\frac{i}{2E}\int_c dt\phi^2} \underbrace{\int D\psi D\bar{\psi} e^{-\int_c dt(\bar{\psi}\partial_t\psi - i\phi\bar{\psi}\psi)}}_{1+\prod_i(1+i\delta_i\phi_i)}. \tag{E.4}$$

Here k labels the time segments on the contour c. If we ignore second order terms and re-exponentiate the product over k we get

$$Z = 1 + e^{-\frac{\beta E}{2}}! \tag{E.5}$$

This is clearly wrong and the problem lies in ignoring second order terms when we re-exponentiate. If, instead of writing

$$\prod_i(1 + i\delta_i\phi_i) \approx \prod_i e^{i\delta_i\phi_i} = e^{\sum_i i\delta_i\phi_i}, \tag{E.6}$$

we write

$$\prod_i(1 + i\delta_i\phi_i) \approx \prod_i e^{i\delta_i\phi_i + \frac{(\delta_i\phi_i)^2}{2}} = e^{\sum_i[i\delta_i\phi_i + \frac{(\delta_i\phi_i)^2}{2}]}, \tag{E.7}$$

which removes the second order terms, we can rectify this problem. However the term δ_i^2 has no meaningful continuum limit.

If we use this form which corrects the second order terms we obtain

$$Z = 1 + \int \frac{D\phi}{\mathcal{N}_\phi} e^{\frac{i}{2E}\sum_i \delta_i(1-iE\delta_i)\phi_i^2 + i\sum_i \delta_i\phi_i}. \tag{E.8}$$

Making the substitution $\phi_i \to \phi_i(1 - iE\delta_i)^{\frac{1}{2}}$ and using

$$\prod_i(1 - iE\delta_i) = e^{\sum_i \ln(1-iE\delta_i)} \approx e^{\sum_i(-\frac{iE\delta_i}{2})} = e^{\frac{E\beta}{2}} \tag{E.9}$$

we find

$$\begin{aligned} Z &= 1 + \prod_i (1 - iE\delta_i) \int \frac{D\phi}{\mathcal{N}_\phi} e^{\frac{i}{2E}\sum_i \delta_i \phi_i^2 + i\sum_i \delta_i \phi_i} && \text{(E.10)} \\ &= 1 + e^{\frac{E\beta}{2}} e^{-\frac{E\beta}{2}} = 2 && \text{(E.11)} \end{aligned}$$

as required.

The problem described above is caused by $\phi_i \sim (E/\delta_i)^{1/2}$, where δ_i is a time segment, hence $(\phi_i\delta_i)^2$ is required to get the terms of order δ_i. We can conveniently fix this problem by considering the interaction in normal ordered form.

Putting the interaction into normal ordered form gives the following interaction to be removed via a Hubbard-Stratonovich transformation (note it must be put into normal ordered form to derive the functional integral expression).

$$\begin{aligned} e^{iS_{int}} &= e^{-i\int dt \frac{E_c}{2} N^2} \\ &\rightarrow e^{-i\frac{E_c}{2}\sum_{i=0}^{N} \delta_i (-\bar{\psi}_{i+1}\bar{\psi}_{i+1}\psi_i\psi_i + \bar{\psi}_{i+1}\psi_i)} && \text{(E.12)} \\ &= \prod_i \int \frac{d\phi_i}{\mathcal{N}_i} e^{-\frac{i\delta_i}{2E_c}\phi_i^2 + (\delta_i\phi_i - \delta_i^2\phi_i^2/2)\bar{\psi}_{i+1}\psi_i} \\ &= \prod_i \int \frac{d\phi_i}{\mathcal{N}_i} e^{-\frac{i\delta_i}{2E_c}\phi_i^2(1 - i\delta_i E_c \bar{\psi}_{i+1}\psi_i) + \delta_i\phi_i\bar{\psi}_{i+1}\psi_i} \\ &= \prod_i e^{\frac{1}{2}\ln(1 - i\delta_i E_c \bar{\psi}_{i+1}\psi_i)} e^{-i\frac{E_c}{2}(\bar{\psi}_{i+1}\psi_i)^2\delta_i} \int \frac{d\phi_i}{\mathcal{N}_i} e^{-\frac{i\delta_i}{2E_c}(\phi_i + iE_c\bar{\psi}_{i+1}\psi_i)^2} \\ &= \prod_i e^{-i\delta_i\frac{E_c}{2}\bar{\psi}_{i+1}\psi_i - i\delta_i\frac{E_c}{2}(\bar{\psi}_{i+1}\psi_i)^2} \quad \text{as required.} && \text{(E.13)} \end{aligned}$$

This gives for the partition function

$$\begin{aligned} \mathcal{Z} &= \int \frac{D\phi}{\mathcal{N}} e^{-\sum_i \frac{i\delta_i\phi_i^2}{2E_c}} \int D\psi e^{-\sum_i \bar{\psi}_{i+1}[1 - i\delta_i\xi - \delta_i\phi_i + \delta_i^2\phi_i^2/2]\psi_i} e^{\bar{\psi}_0\psi_0 - \sum_{i=1}^{N-1}\bar{\psi}_i\psi_i} \\ &= \int \frac{D\phi}{\mathcal{N}} e^{-\sum_i \frac{i\delta_i\phi_i^2}{2E_c}} \prod_k [1 + \Pi_i(1 - i\delta_i\xi_k + \delta_i\phi_i - \delta_i^2\phi_i^2/2)] \\ &= \int \frac{D\phi}{\mathcal{N}} e^{-\sum_i \frac{i\delta_i\phi_i^2}{2E_c}} \prod_k [1 + e^{\int_c dt(\phi - i\xi_k)}]. && \text{(E.14)} \end{aligned}$$

The sum over the state label k is suppressed in the first line for clarity. Notice that, as the fermionic integral gives an expression of second order, we can re-exponentiate without causing any problems.

Appendix F

GAUGE TRANSFORMATIONS IN FUNCTIONAL INTEGRALS

The following is a demonstration, in both the continuous and discrete representations, of why it is not possible to "gauge out" the entire Bosonic field in the one dimensional problem. No such issue arises in, for example, the one dimensional system of a Luttinger Liquid.

F.1 Jacobian for Grassmann Variables

Due to the equivalence of integration and differentiation with Grassmann variables the Jacobian associated with a change of Grassmann variables in an integral is

$$J^{-1} = \left| \mathrm{Det}\left(\frac{\partial \psi_i}{\partial \psi'_j} \right) \right|. \tag{F.1}$$

Where the change of variables is $\{\psi_i\} \to \{\psi'_i\}$. This can be easily proved by induction.

Now, where we have two independent sets of fields, as in the functional integrals we are looking at, we will find

$$\int D\psi D\bar{\psi} \ldots = \int D\psi' D\bar{\psi}' J \ldots \tag{F.2}$$

$$J^{-1} = \left| \mathrm{Det}\left(\frac{\partial \psi_i}{\partial \psi'_j} \right) \right| \left| \mathrm{Det}\left(\frac{\partial \bar{\psi}_i}{\partial \bar{\psi}'_j} \right) \right| = \prod_i [e^{-\Im\theta_i - \Im\chi_i}] \text{ for} \tag{F.3}$$

$$\psi_i = \psi'_i e^{i\theta_i} \text{ and} \tag{F.4}$$

$$\bar{\psi}_i = \bar{\psi}'_i e^{i\chi_i}. \tag{F.5}$$

For any instance we will be interested in $\chi_i = -\theta_i$ and the Jacobian will be 1. All other possibilities will give highly complicated actions and we wish the gauge field to cancel from the action (as far as possible).

F.2 The Discrete Case

First let us look at the issue in the discrete notation. We have the following action

$$\bar{\psi}_0\psi_0 - \sum_{i=1}^{N} \bar{\psi}_i\psi_i + \sum_{i=0}^{N} \bar{\psi}_{i+1}\psi_i[1 - i(\xi + i\phi_i)\delta_i] \tag{F.6}$$

where the field ψ_{N+1} is defined as $-\psi_0$ so that we can write a continuous notation (and similarly for $\bar{\psi}_{N+1}$.) It is important that we only really have the fields from $i = 0 \dots N$. Making the gauge transformation $\psi_i \to \psi_i e^{i\theta_i}$ and $\bar{\psi}_i \to \bar{\psi}_i e^{-i\theta_i}$ we have

$$\bar{\psi}_0\psi_0 - \sum_{i=1}^{N} \bar{\psi}_i\psi_i + \sum_{i=0}^{N} \bar{\psi}_{i+1}\psi_i[1 - i(\xi + i\phi_i)\delta_i]e^{i\theta_i - i\theta_{i+1}}. \tag{F.7}$$

Assuming θ is a smooth function we can expand the exponent $e^{i\theta_i - i\theta_{i+1}}$ and, demanding $i\theta_i - i\theta_{i+1} \sim \delta_i$, we get the following condition for removing the field ϕ:

$$\frac{\theta_i - \theta_{i+1}}{\delta_i} = -i\phi_i. \tag{F.8}$$

We require a smooth function θ which satisfies this condition. Additionally we have a term θ_{N+1} which we would like to satisfy some boundary conditions. This is because the term is introduced only to define equation(F.8) properly. When the field θ is introduced it is with $\psi_{0\dots N}$ but there is no actual ψ_{N+1}. Let $\theta_{N+1} = \theta_0(+2\pi n) + \theta'$, if $\theta' = 0$ the boundary condition $\psi_{N+1} = -\psi_0$ will be satisfied but it may not be possible to find a field θ which does this. Note from the above that $\theta'(+2\pi n) = i\sum_i \delta_i\phi_i$.

We now have, in discrete notation,

$$\bar{\psi}_0\psi_0 - \sum_{i=1}^{N} \bar{\psi}_i\psi_i + \sum_{i=0}^{N-1} \bar{\psi}_{i+1}\psi_i[1 - i(\xi + i\phi_i)\delta_i]e^{i\theta_i - i\theta_{i+1}} - \bar{\psi}_0\psi_N[1 - i(\xi + i\phi_i)\delta_i]e^{i\theta_N - i\theta_0 - i\theta'} \tag{F.9}$$

which gives

$$\bar{\psi}_0\psi_0 - \sum_{i=1}^{N} \bar{\psi}_i\psi_i + \sum_{i=0}^{N-1} \bar{\psi}_{i+1}\psi_i[1 - i\xi\delta_i] - \bar{\psi}_0\psi_N[1 - i\xi\delta_i]e^{-i\theta'}. \tag{F.10}$$

So we have gauged out the field except for one anomalous term. Were this term to be zero then clearly the bosonic field ϕ would be completely separable from the fermionic action (and would not appear in the partition function at all). An analogous calculation can be performed in the continuous notation. This shall be done in the next section.

The anomalous term can be included in the discrete calculations done previously in section 2.4.3 leaving

$$\begin{aligned}
Z &= -\int (D^{N+1}\psi)e^{-\bar{\psi}D\psi} && \text{(F.11)}\\
&= \det D = 1 + \prod_{k=1}^{N+1} a_k e^{-i\theta'} && \text{(F.12)}\\
&= 1 + e^{-i\int_c dt\xi - i\theta'} = 1 + e^{\int_c dt(\phi - i\xi)} && \text{(F.13)}
\end{aligned}$$

as before (suppressing the product over states m of ξ_m).

For the Green's function we can do the same:

$$\begin{aligned}
iG(t,t') &= -e^{i\theta_i - i\theta_j} \int (D^{N+1}\psi)e^{-\bar{\psi}D\psi}\psi_i\bar{\psi}_j && \text{(F.14)}\\
&= e^{i\theta_i - i\theta_j} \begin{cases} \displaystyle\prod_{k=j}^{i+1} a_k & \text{if } i > j \\ \displaystyle -\prod_{k=1}^{i-1}\prod_{j}^{N+1} a_k e^{-i\theta'} & \text{if } i < j \end{cases} && \text{(F.15)}\\
&= e^{i\theta(t) - i\theta(t')} \begin{cases} e^{-i\int_{t'}^{t}\xi dt''} & \text{if } t > t' \\ -e^{-i\int_{t_0}^{t}\xi dt'' - i\int_{t'}^{t_{N+1}}\xi dt''} e^{-i\theta'} & \text{if } t < t' \end{cases} && \text{(F.16)}
\end{aligned}$$

as before.

F.3 Boundary Conditions in the Continuous Form

If we wish to make a gauge transformation in

$$i\int_c dt\bar{\psi}[i\partial_t - \xi - i\phi]\psi \tag{F.17}$$

we need to satisfy the anti-periodic boundary conditions of the fermionic fields which are an integral part of this representation. Let $\psi \to \psi e^{i\theta(t)}$ and $\bar{\psi} \to \bar{\psi}e^{-i\theta(t)}$. The anti-periodic boundary conditions enforce the condition $\theta(t_0 - i\beta) - \theta(t_0) = 2\pi n$. Now, for

$$\partial_t\theta = \phi \quad \text{and} \quad i\partial_t g(t,t') = \delta(t-t') \tag{F.18}$$

$$\theta(t) = \int_c dt' g(t,t')\phi(t') \tag{F.19}$$

to satisfy the boundary conditions, we need $ig(t,t')$ to obey bosonic periodic boundary conditions on the contour. For a case where these conditions can be satisfied, the field ϕ can be gauged out of the action entirely, this is equivalent to $\theta' = 0$ for the discrete case.

For our problem this is not possible, as the solution to $i\partial_t g(t,t') = \delta(t-t')$ is a step function (plus a constant) defined on the contour. No step function obeys periodic boundary conditions and hence no solution exists. However, following the transformation made by Kamenev and Gefen[24], we can perform the following

$$\psi \to \psi e^{\int^t dt'[\phi(t')-iT\phi_0]} \text{ and } \bar{\psi} \to \bar{\psi}e^{\int^t dt'[\phi(t')-iT\phi_0]} \tag{F.20}$$

which will gauge out all but the "zero-mode" of the field ϕ, $\phi_0 = \int_c dt\phi$. This transformation obeys the boundary conditions of ψ. In the same way as for the discrete case we then have

$$Z = \int D\psi e^{i\int_c dt\bar{\psi}[i\partial_t - \xi + T\phi_0]\psi}. \tag{F.21}$$

Note that from the form of equation(F.13) it is clear we can not separate out the zero-mode of the field in any trivial way.

Appendix G

THE LINDHARD FUNCTION

We can define the Lindhard function in Keldysh as

$$\Pi^{kk'}(x-x') = 2i\tilde{\gamma}^{k}_{j'i'}G_{i'i}(x-x')G_{jj'}(x'-x)\gamma^{k'}_{ij}. \tag{G.1}$$

It is the free particle "bubble" or polarization operator. We shall refrain from referring to it as the polarization operator and call it the Lindhard function to avoid confusion with the disordered case. From this we find

$$\begin{aligned}
\Pi^R(\omega,\mathbf{q}) &= \frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\int\frac{d\varepsilon}{2\pi}\big\{G^R(P)G^A(P+Q)[h(\varepsilon)-h(\varepsilon+\omega)] \\
&\qquad +G^R(P)G^R(P+Q)h(\varepsilon+\omega)-G^A(P)G^A(P+Q)h(\varepsilon)\big\} && \text{(G.2)}\\
\Pi^A(\omega,\mathbf{q}) &= [\Pi^R(\omega,\mathbf{q})]^* && \text{(G.3)}\\
\Pi^{21}(\omega,\mathbf{q}) &= 0 && \text{(G.4)}\\
\Pi^K(\omega,\mathbf{q}) &= \frac{i}{2}\int\frac{d^dp}{(2\pi)^d}\int\frac{d\varepsilon}{2\pi}[h(\varepsilon)h(\varepsilon+\omega)-1](G^R_\varepsilon(\mathbf{p})-G^A_\varepsilon(\mathbf{p})) \\
&\qquad\qquad \times(G^R(Q+P)-G^A(Q+P)) \\
&= \coth\left(\frac{\omega}{2T}\right)\big[\Pi^R(\mathbf{q},\omega)-\Pi^A(\mathbf{q},\omega)\big]. && \text{(G.5)}
\end{aligned}$$

For low enough temperatures $h(x) \approx \mathrm{sgn}(x)$ and the frequency integration can be trivially dealt with. For the last two terms in equation (G.2) we are only interested in the lowest order in ω and $\mathbf{q}$. Using the fact that $h_\omega = 1 - 2f_\omega$, where f is the Fermi function, we can write

$$-2i\int\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{0}\frac{d\varepsilon}{2\pi}[(G^R(\mathbf{p},\varepsilon))^2-(G^A(\mathbf{p},\varepsilon))^2] = \frac{i}{\pi}\int\frac{d^3p}{(2\pi)^3}[G^R(\mathbf{p},\varepsilon)-G^A(\mathbf{p},\varepsilon)]\Big|^{0}_{-\infty} \tag{G.6}$$

After momentum integration we are left with $-2\nu_0$. The first terms in equation (G.2) have to be dealt with slightly more carefully. After linearizing ξ

$$\Pi^R(\omega, \mathbf{q}) = -2\nu_0 - i\nu_0 \int d\xi \int_{-1}^{1} d\Omega \int_{-\omega}^{0} \frac{d\varepsilon}{2\pi} \frac{1}{\varepsilon - \xi + i\delta} \frac{1}{\varepsilon + \omega - \xi - v_F q\Omega - i\delta}. \quad \text{(G.7)}$$

The frequency integral is simply performed after rescaling ξ. This integral can then also be done leaving us

$$\begin{aligned}\Pi^R(\omega, \mathbf{q}) &= -2\nu_0 - \nu_0\omega \int_{-1}^{1} d\Omega \frac{1}{\omega - v_F q\Omega - 2i\delta} \\ &= -2\nu_0 + \frac{\nu_0\omega}{v_F q} \ln\left[\frac{1 - \omega/v_F q + i\delta}{-1 - \omega/v_F q - i\delta}\right]. \end{aligned} \quad \text{(G.8)}$$

In the limit $|\omega| \ll v_F q$ we find

$$\Pi^R(\omega, \mathbf{q}) \approx -2\nu_0\left[1 - \frac{i\pi\omega}{2v_F q} - \frac{1}{3}\frac{q^2}{4p_F^2}\right]. \quad \text{(G.9)}$$

The q^2 term originates from higher order terms. For a full calculation see Mahan p395[70].

The Lindhard function is defined in the Matsubara representation as

$$\chi_0(\mathbf{q}, \omega_n) = -T \sum_{k,\epsilon_m} G(\mathbf{k}, \epsilon_m) G(\mathbf{k} + \mathbf{q}, \omega_n + \epsilon_m). \quad \text{(G.10)}$$

ϵ_m are the fermionic, and ω_n the bosonic, Matsubara frequencies. Using an appropriate contour

$$\oint_c d(i\varepsilon) f(i\varepsilon) g(i\varepsilon) = 2\pi i T \sum_{\substack{\varepsilon_n = \\ (2n+1)\pi T}} g(i\varepsilon), \text{ where} \quad \text{(G.11)}$$

$$f(i\varepsilon) = \frac{1}{e^{i\beta\varepsilon} + 1}. \quad \text{(G.12)}$$

Hence we can rewrite

$$\chi_0(\mathbf{q}, i\omega_n) = \frac{1}{2\pi i} \sum_{\mathbf{k}} \oint_c d\varepsilon f(\varepsilon) \frac{1}{\varepsilon - \xi_{\mathbf{k}}} \frac{1}{\varepsilon + i\omega - \xi_{\mathbf{k}+\mathbf{q}}}. \quad \text{(G.13)}$$

We have introduced $\varepsilon = i\epsilon$ and $\omega = i\omega$. We can now deform the contour so the part on the negative axis cancels and the part on the positive real axis encircles the two poles. Hence

$$\chi_0(\mathbf{q},\omega) = -\sum_{\mathbf{k}}\left[\frac{f(\xi_{\mathbf{k}})}{\xi_{\mathbf{k}}+\omega-\xi_{\mathbf{k+q}}}+\frac{f(\xi_{\mathbf{k+q}}-\omega)}{\xi_{\mathbf{k+q}}-\xi_{\mathbf{k}}-\omega}\right] \tag{G.14}$$

$$= \sum_{\mathbf{k}}\frac{f_{\mathbf{k}+\frac{\mathbf{q}}{2}}-f_{\mathbf{k}-\frac{\mathbf{q}}{2}}}{\omega-\frac{\mathbf{k}.\mathbf{q}}{m}}. \tag{G.15}$$

This can be evaluated exactly but we are interested only in its form for $\omega \ll v_F q$ and $q \ll k_F$. At zero temperature and in three dimensions we find (similarly to the Keldysh case)

$$\chi_0(\mathbf{q},\omega_n) \approx \nu_0\left[1-\frac{1}{3}\left(\frac{q}{2p_F}\right)^2-\frac{\pi}{2}\left(\frac{|\omega_n|}{qv_F}\right)\right]. \tag{G.16}$$

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